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ABSTRACT

This text provides a one-semester study of analytic geometry for secondary school students. It is designed for use at the 12th grade level. A deliberate effort was made to tie this text to previous SMSG texts; the usual language of sets, ordered pairs, number properties, etc. are included. This flavor is what distinguishes this book from others in the field. Ten chapters included in the book are: (1) Analytic Geometry; (2) Coordinates and the Line; (3) Vectors and their Applications; (4) Proofs by Analytic Methods; (5) Graphs and their Equations; (6) Curve Sketching and Locus Problems; (7) Conic Sections; (8) The Line and the Plane in 3-Space; (9) Quadric Surfaces; and (10) Geometric Transformations. Also included are an index and a section of supplements to the various chapters. (RH)

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Analytic Geometry

Student's Text

REVISED EDITION

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PREFACE

This text aims at restoring what is, in a sense, a "lost" subject. There is a widespread practice of including analytic geometry material in the calculus program; but when this is accomplished, Analytic Geometry, as a course of study, disappears and what remains of it is the part immediately useful to a study of calculus. You will find a much more varied selection of topics in this book than you would see in a calculus course.

In a book devoted to the interplay between algebra and geometry you would expect to be called upon to exhibit considerable dexterity in algebraic manipulations as well as to recall previous experiences with geometric figures and theorems. You will not be disappointed. It is also assumed that you know the elementary notions of trigonometry.

A deliberate effort was made to tie this text to previous SMSG texts; so, you will find the usual language of sets, ordered pairs, number properties, etc., with which you have had some acquaintance. This flavor is perhaps what distinguishes this book from others in the field. For example, the treatment of coordinate systems in Chapter 2 depends upon the postulates of SMSG Geometry.

Here is one word of advice. The early chapters are fundamental to everything which follows. Study them until they seem to be old friends; do not hesitate to return to them later for a fresh look. Another thing you might watch. The related ideas of vectors, direction numbers, and parameters are used extensively to simplify and unify the various topics. Look for this feature.

The theorems and figures are numbered serially within each chapter; e.g., Theorem 8-3 is the third theorem of Chapter 8, Figure 5-2 is the second figure to appear in Chapter 5. If an equation is to be referred to, it is assigned a counting number, which is then displayed in the left margin. The counting begins at one for each section. Definitions are not numbered but may be found by referring to the Index.

The writers hope they have recreated the beauty of Analytic Geometry in a new SMSG setting, and they further hope that you will enjoy and profit by the adventure you are about to undertake. Bon Voyage.

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Chapter 1

ANALYTIC GEOMETRY

1-1. What Is Analytic Geometry?

Geometry has been studied systematically for over two thousand years. Euclid's Elements, which was written about 300 B.C., is perhaps the most influential mathematics textbook ever published. There are undoubtedly many traces of it to be found in the text you used in your high school course.

Until the 17th century, geometry was studied by what are known as synthetic methods. The postulates dealt with such geometric notions as point, line, and angle, and little or no use was made of numbers. In the Elements, for example, line segments do not have lengths.

Then in the early part of the 17th century there occurred the greatest advance in geometry since Euclid. It was not the work of one man--such advances seldom, if ever, are. Instead, it occurred when the "intellectual climate" was ready for it. Nevertheless, there was one man whose name is so universally associated with the new geometry that you should know it. That man was René Descartes, a French mathematician and philosopher, who lived from 1596 to 1650. The essential novelty in the new geometry was that it used algebraic methods to solve geometric problems. Thus it brought together two subjects which until then had remained almost independent.

The link between geometry and algebra is forged by coordinate systems. In essence, a coordinate system is a correspondence between the points of some "space" and certain ordered sets of numbers. (We use quotation marks because the space may be a curve, or the surface of a sphere, or some other set of points not usually thought of as a space.) You are already familiar with a number of different coordinate systems, some studied in earlier mathematics courses, others met with in other fields, such as geography. In elementary algebra you introduced coordinates into a plane by drawing two mutually perpendicular lines (axes) in the plane, choosing a positive direction on each and a unit length common to both, and associating with each point the ordered pair of real numbers representing the directed distances of the point from the two axes. The location of a point on the earth's surface is often given in terms of latitude and longitude. An artilleryman sometimes locates a target by saying how far away it is, and in what direction it lies with respect to an arbitrary fixed direction established by setting up an aiming post. This is what is called a polar coordinate system for the plane.

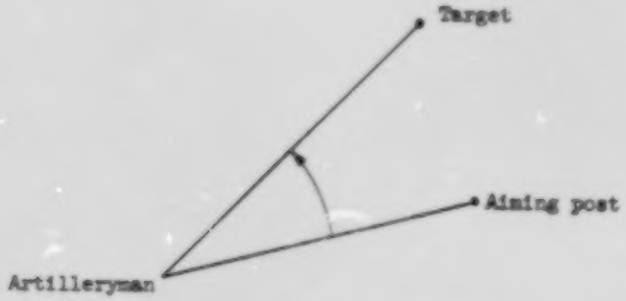


Figure 1-1

A point P on a right circular cylinder could be identified by means of the directed distance z and the measure of the angle θ shown in Figure 1-2.

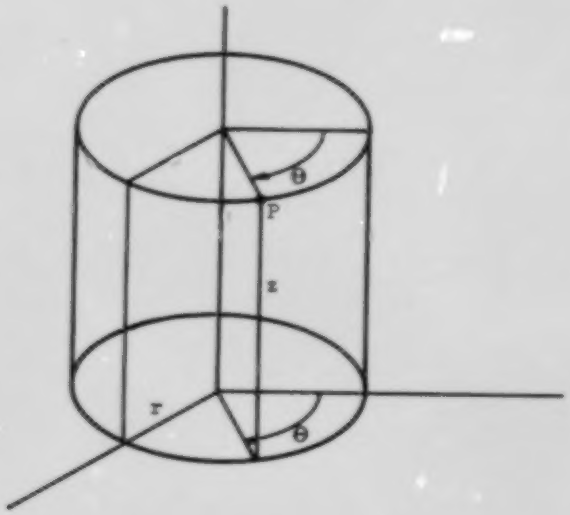


Figure 1-2

If, instead of one right circular cylinder, we consider all such cylinders with the same axis, we can locate any point in space by giving the radius r of the cylinder on which it lies and its z- and θ- coordinates on that cylinder. The result is called a cylindrical coordinate system for space.

A fly on a doughnut (a point on a torus) could be located by means of the measures (in degrees, radians, or any other convenient unit) of the angles θ and Φ shown in the figure below.

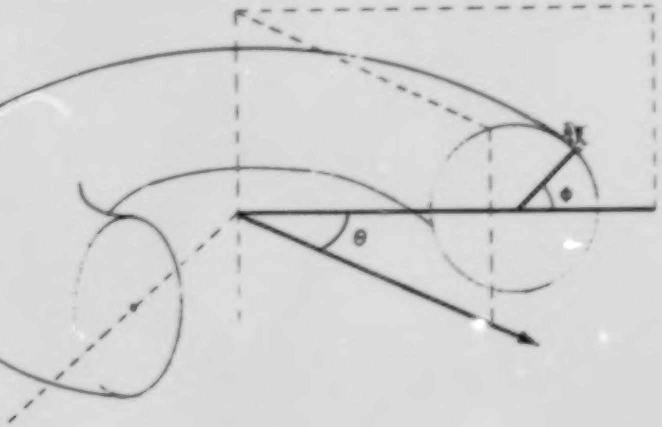


Figure 1-3

The position of an artificial satellite at a certain moment could be specified by giving its vertical distance from the earth's surface (or center) and the latitude and longitude of the point of the earth's surface directly "below" the satellite.

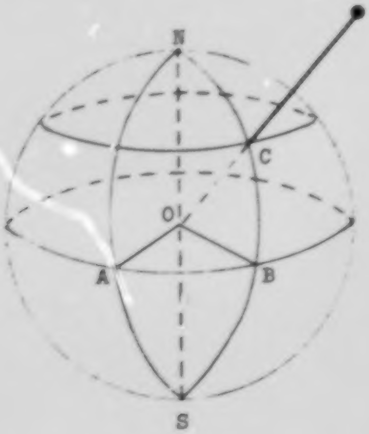


Figure 1-4

The result is called a spherical coordinate system for space.

A coordinate system could be set up even for a "space" which is quite irregular. We may note that your home address is a set of coordinates with which we locate a particular point, your home, relative to the streets and avenues of the town you live in. These streets and avenues, which need not be straight, are the "coordinate lines", and the numbers of the houses on them indicate, in some reasonable way, the positions along these lines.

Once a coordinate system has been established, interesting sets of points can be represented by suitable conditions on their coordinates. The equation

$$2x - y + 4 = 0$$

represents the line through the points $(-1,2)$ and $(2,8)$, where we are using rectangular coordinates. The inequality

$$x^2 + (y - 2)^2 < 9$$

represents the set of points not as far as 3 units distant from $(0,2)$, in other words, the interior of the circle with radius 3 and center $(0,2)$. The equation

$$x^2 - y^2 = 0$$

represents the two lines through the origin making angles of 45° and 135° with the x-axis.

By means of coordinate systems we can, if you like, arithmetize geometry. Problems about geometric figures are replaced by problems about numbers, functions, equations, inequalities, and so forth. Thus one can bring to bear the extensive body of knowledge about algebra, trigonometry, and the calculus which has been developed largely since the 13th century. (In this text we shall use no calculus, but if later you study the subject, you will see that it would have been, in some places, rather useful to us.)

The definition of analytic geometry given above is of the sort found in dictionaries rather than the sort used in mathematics. It tells us not how a technical term will be used in the remainder of this book but how a non-technical phrase is commonly used. As the discussion above indicates, both the subject matter and the methods of this book are already fairly familiar to you. You have even put them together in earlier courses. For example, you know that the graph (in a plane) of an equation of the form

$$ax + by + c = 0 \tag{1}$$

is a straight line, and that the problem of finding the intersection of two lines in a plane can be solved by finding the solution of a system of two equations like (1). You also know that the locus of all the points in a plane which are as far from a fixed line as they are from a fixed point not on that line (this is called a parabola) has an equation of the form

$$y^2 = 4cx$$

if you choose the proper coordinate system. In this book we shall take up many such problems, and by the time you reach the end of it you will have some idea of the power of the new method which Descartes and his contemporaries introduced into geometry.

1-2. Why Study Analytic Geometry?

A chief reason for studying analytic geometry is the power of its methods. Certain problems can be solved more readily, more directly, and more simply by such methods. This is true not only for the problems of geometry and other branches of mathematics, but also for a wide variety of applications in statistics, physics, engineering, and other scientific and technical fields.

Using algebraic methods to solve geometric problems permits easy generalization. A result obtained in one or two dimensions can often be extended at once to three or more dimensions. It is often just as easy to prove a relation in space of n dimensions as it would be in space of two or three dimensions. In fact, much of the work in higher dimensions is essentially algebra with geometric terminology.

Analytic geometry ties together and applies in a new and interesting context what you have been learning about number systems, algebra, geometry, and trigonometry. It should lead to mastery in handling mathematics you have studied previously. As you study this course you will have many opportunities to use knowledge and methods that constitute your present mathematical equipment. You will also learn new methods. Sometimes the new methods will seem awkward or difficult at first when compared with methods you have been using. You should keep in mind that what you are doing is learning about the methods and how to apply them.

As a student, you may at times be directed to use a certain method to gain facility with it. Real problems, whether in mathematics, science, or industry, do not come equipped with a mathematical setting and a prescribed method. By the end of this course you should have a greater variety of mathematical weapons in your arsenal, and more powerful ones. You should be more able then to select effective mathematical weapons to attack problems. Thus another important reason for studying analytic geometry is the value it will have for you in future courses--not just courses in mathematics but in physics, statistics, engineering, and science in general.

There is a current trend to combine analytic geometry and calculus. When this occurs, much that is of value in the subject of analytic geometry is lost. Because such a course is primarily calculus, only such parts of analytic geometry as are immediately useful in the calculus are kept. By studying a separate course in analytic geometry, you have a better opportunity to understand the coherence of the subject, the diversity of its methods, and the wide variety of problems to which it may be applied.

One of the most important reasons for studying analytic geometry is to gain understanding of the interplay of algebra and geometry. Algebra contributes to analytic geometry by providing a way of writing relationships, a method not only of proving known results but also of deriving previously unknown results. Geometry contributes to algebra by providing a way of visualizing algebraic relations. This visualization, or picture, helps you to understand the algebraic discussion. In the framework provided by a coordinate system, you will do geometry by doing algebra, and see algebra by looking at geometry. Algebra and geometry are intermeshed in analytic geometry; each strengthens and illuminates the other.

Chapter 2

COORDINATES AND THE LINE

2-1. Linear Coordinate Systems.

In our previous study of mathematics we have already encountered at least three major mathematical structures, arithmetic, the algebra of real numbers, and Euclidean geometry. The great German mathematician, David Hilbert (1862-1943), showed that all geometric problems could be reduced to problems in algebra. Our goal here need not be so drastic. We are not trying to eliminate the need for geometry, but rather to establish connections between algebra and geometry. This will enable us to bring to bear on a single problem both the power of algebraic techniques and the structural clarity of geometry.

It turns out that we are able to effect these connections between algebra and geometry by establishing certain one-to-one correspondences between real numbers and points on a line and between real numbers and angles.

In our study of geometry we adopted an important postulate:

> The Ruler Postulate. The points of a line can be placed in correspondence with the real numbers in such a way that
>
> (1) To every point of the line there corresponds exactly one real number,
>
> (2) To every real number there corresponds exactly one point of the line, and
>
> (3) The distance between two points is the absolute value of the difference of the corresponding numbers.

We defined such a correspondence to be a coordinate system for the line. We called the number corresponding to a given point the coordinate of the point.

In order to assign a coordinate system to a given line we adopted another postulate:

> The Ruler Placement Postulate. Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

We found these postulates to be extremely useful when we defined such concepts as congruence for segments, and order or betweenness for collinear points. We shall want to review and extend these ideas in this text, for it is through coordinate systems that we are able to relate the algebra of numbers to the geometry of sets of points. We shall first extend our notion of a coordinate system.

In our theoretical development of geometry we had no need to mention units; the measure of distance between each pair of points was always a fixed, though unspecified, number. We did not need to know what these numbers were, but only how the measure of distance between one pair of points compared with the measure of distance between a second pair of points. Was the first number as large as the second? Was it larger? Was it twice as large? In applying our theoretical knowledge to specific problems we found that we could use any units we pleased if we were consistent in our usage throughout each given problem. If we did a problem in inches rather than in feet, the numbers we obtained were twelve times as great, but equal distances were still measured by equal numbers. A greater distance had a greater measure, and a shorter distance had a smaller measure, but the ratio of these distances was the same for both choices of unit. Although the measures of distance between pairs of points depended upon the choice of units, within a given problem the measures in one unit were always proportional to the corresponding measures in another unit.

What we discovered in effect was that relative to a given point on a line there are not just two coordinate systems for the line, one oriented in each direction. For each point and each sense of direction on the line there is a coordinate system for the line corresponding to each choice of unit for measuring distance. In each of these coordinate systems the orientation corresponds to one sense of direction for the line and the coordinate of the given point is zero. Since there are infinitely many choices of unit, there are infinitely many coordinate systems for each point and sense of direction on the line.

In this text we are not attempting to develop a rigorous deductive system as we did in geometry. Rather we want to develop and extend the concepts and techniques which we can use to solve problems. Our basic technique will be to introduce coordinate systems. It is so important to utilize the freedom to choose coordinate systems on a line that we state the following guiding principle:

LINEAR COORDINATE SYSTEM PRINCIPLE. There exist coordinate systems for any line such that:

(1) If P and Q are any two distinct points on the line and p and q are any two distinct real numbers, there is a coordinate system in which the coordinate of P is p and the coordinate of Q is q.

(2) If P, Q, R, and S are collinear points with coordinates p, q, r, and s respectively in one coordinate system and p', q', r', and s' respectively in a second coordinate system, if P and Q are distinct, and if R and S are distinct, then

$$\frac{|p' - q'|}{|p - q|} = \frac{|r' - s'|}{|r - s|} .$$

DEFINITION. If a coordinate system on a line assigns the coordinates r and s to the points R and S, then $|r - s|$ is the *measure of distance between* R *and* S *relative to the coordinate system.*

This nicety of expression is necessary when we are trying to explain and distinguish concepts which are often confused. As our understanding increases, we may speak more colloquially, and use whatever level of precision is appropriate to the topic and setting. What is important is that a lack of precision should reflect our choice and not our ignorance.

For convenience, and if there is no danger of ambiguity, we shall call this the *distance* between R and S.

We denote the distance between R and S by $d(R,S)$.

Wherever the context makes clear that only a single coordinate system is being considered, we shall adopt the convention that a is the coordinate point of A, b is the coordinate of point B, c is the coordinate of point C, ... We shall call the point with coordinate zero the *origin* of the coordinate system. The point with coordinate one is called the *unit-point*.

It is sometimes convenient to think of the *directed distance from* R *to* S, which we define to be the number $s - r$. We shall need this idea in the next section.

We shall also find it necessary to use the notion of a *directed segment*, which we define to be the set whose elements are the segment and the ordered pair of its endpoints, or $\{\overline{RS},(R,S)\}$. We shall denote such a directed segment by $\overrightarrow{RS}$. The directed segment $\overrightarrow{RS}$ is said to *emanate* from R and *terminate* in S. However, we should note that directed distance is related to the choice of coordinate system and a directed segment is related to the choice of order for its endpoints. The *length* or *magnitude* of the directed segment $\overrightarrow{RS}$ is the length of $\overline{RS}$, or $d(R,S)$. The ordering of the pair of endpoints (R,S) is related to our intuitive notion of sense of direction, from R to S. We shall find that this alliance of the concepts of magnitude and sense of direction in directed segments is basic to our development of a powerful tool of analysis in Chapter 3.

We conclude with two examples illustrating some of the ideas introduced above.

Example 1. Let us perform a practical experiment. Take a ruler which is marked in inches and another which is marked in centimeters; use each of these rulers to measure the distances between the pairs of labeled points in Figure 2-1. Record your results and compare them.

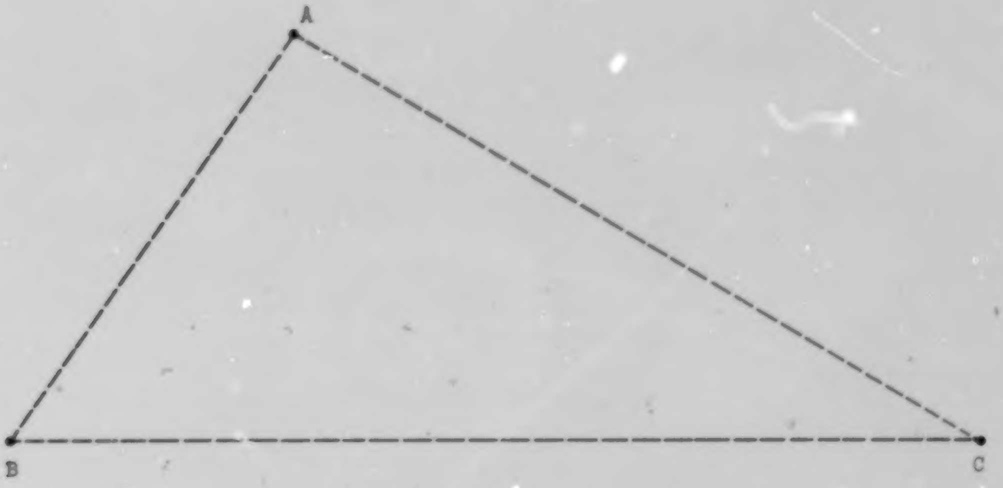


Figure 2-1

Discussion. If a ruler is old or damaged at an end, we prefer not to measure from the end. When we made the measurements required above, we happened to place the unit point of the coordinate system on the inch ruler at A and found that in this case the coordinates of B and C were $3\frac{7}{8}$ and $5\frac{5}{8}$ respectively. When we placed the unit point at B, we found the coordinate of C to be $6\frac{5}{8}$. Since the measure of distance is the absolute value of the difference between the coordinates, we concluded that in inches $d(A,B) = 2\frac{7}{8}$, $d(A,C) = 4\frac{5}{8}$, and $d(B,C) = 5\frac{5}{8}$. We made similar measurements using a ruler marked in centimeter units.

We summarized our measurements in the following table.

Distance	Measure in inches	Measure in centimeters
d(A,B)	$2\frac{7}{8}$	7.3
d(A,C)	$4\frac{5}{8}$	11.7
d(B,C)	$5\frac{5}{8}$	14.3

How do these results compare with yours?

We compared the measures to each other, first in inches and then in centimeters:

$$\frac{d(A,B)}{d(A,C)} = \frac{2\frac{7}{8}}{4\frac{5}{8}} \approx .62 \quad , \qquad \frac{d(A,B)}{d(A,C)} = \frac{7.3}{11.7} \approx .62 \; ,$$

$$\frac{d(A,B)}{d(B,C)} = \frac{2\frac{7}{8}}{5\frac{5}{8}} \approx .51 \quad , \qquad \frac{d(A,B)}{d(B,C)} = \frac{7.3}{14.3} \approx .51 \; ,$$

$$\frac{d(A,C)}{d(B,C)} = \frac{4\frac{5}{8}}{5\frac{5}{8}} \approx .82 \quad , \qquad \frac{d(A,C)}{d(B,C)} = \frac{11.7}{14.3} \approx .82 \; .$$

The accuracy of our results cannot exceed that of our measurements. Within these limitations we found that the ratios of corresponding measures of distance were independent of the units.

Then we compared the measurements in centimeters to those in inches for the same pairs of points and for the perimeter of $\triangle ABC$:

$$d(A,B) : \frac{7.3}{2\frac{7}{8}} \approx 2.54 \; ,$$

$$d(A,C) : \frac{11.7}{4\frac{5}{8}} \approx 2.53 \; ,$$

$$d(B,C) : \frac{14.3}{5\frac{5}{8}} = 2.54 ,$$

$$\text{Perimeter of } \triangle ABC : \frac{33.3}{13\frac{1}{8}} = 2.54 .$$

Within the limits of accuracy which we could expect, we found that the corresponding measurements in centimeters and in inches were proportional.

Example 2. A straight road 180 miles long connects town A to town B . A driver leaves town A for B at the same instant as another driver leaves town B for A . The drivers travel at the uniform rates of speed, 44 ft. per sec. and 88 ft. per sec. respectively. How soon will they meet?

Discussion. In solving this problem we must make some decisions about units. Some information is given in terms of miles and some in terms of feet. Also we are not told in what units to express the answer. Suppose we try two different approaches. We shall first adopt feet and seconds as the units for distance and time.

(1) We must express 180 miles in feet. The constant of proportionality is 5,280 ft. per mile.
Thus

$$180 \ (\text{mi.}) \times \frac{5280}{1} \left(\frac{\text{ft.}}{\text{mi.}}\right) = 950,400 \ (\text{ft.}) .$$

The inclusion of the name of the unit next to the number of units is a common practice in the physical sciences and engineering. It provides an immediate reminder of the significance of the calculations. Such a practice is called a mnemonic (from the Greek μνασθαι meaning to remember).

We let t represent the number of seconds which will elapse before the two drivers meet. We interpret the problem with the following statement of equality:

$$44t + 88t = 950,400 ,$$

which is equivalent to

$$132t = 950,400$$

and

$$t = 7,200 .$$

The drivers will meet in 7,200 seconds.

This result is such a large number that it may not appeal readily to our intuitive sense of duration of time. We might convert this measure to different units in the hope that the answer will be more intuitively meaningful. If we convert to minutes by dividing by 60, we obtain 120 minutes, which is clearer. If we convert to hours by dividing by 60 again, we obtain 2 hours, which is probably the most satisfactory expression of the answer.

If we are able to anticipate the relative size of the answer, we may be able to choose units which will obviate the need to make changes at the end. In this problem we might well have realized that hours were an appropriate unit for time. We might also have simplified the arithmetic had we used miles as the unit of distance. Our solution would then have been:

(2) We convert the rates of speed to miles per hour. The constants of proportionality are $\frac{1}{5280}$ mile per foot, 60 seconds per minute, and 60 minutes per hour. Thus we obtain

$$44\ \left(\frac{\text{ft.}}{\text{sec.}}\right) \times \frac{1}{5280}\ \left(\frac{\text{mi.}}{\text{ft.}}\right) \times \frac{60}{1}\ \left(\frac{\text{sec.}}{\text{min.}}\right) \times \frac{60}{1}\ \left(\frac{\text{min.}}{\text{hr.}}\right) = 30\ \left(\frac{\text{mi.}}{\text{hr.}}\right)$$

and

$$88\ \left(\frac{\text{ft.}}{\text{sec.}}\right) \times \frac{1}{5280}\ \left(\frac{\text{mi.}}{\text{ft.}}\right) \times \frac{60}{1}\ \left(\frac{\text{sec.}}{\text{min.}}\right) \times \frac{60}{1}\ \left(\frac{\text{min.}}{\text{hr.}}\right) = 60\ \left(\frac{\text{mi.}}{\text{hr.}}\right).$$

We let t represent the number of hours which will elapse before the two drivers meet. We interpret the problem with the statement of equality,

$$30t + 60t = 180.$$

This is equivalent to

$$90t = 180$$

or

$$t = 2.$$

The drivers will meet in 2 hours.

The first example illustrates the assertions which led to the formulation of the Linear Coordinate System Principle. It also suggests that when we change the coordinate system, we do not lose the notion of congruence for segments, which is defined in the SMSG Geometry on the basis of equal lengths. In the next section we shall see that the concept of order or betweenness is also preserved in linear coordinate systems.

The second example points up the necessity for using units consistently throughout the solution of a problem. It also illustrates the advantages inherent in the freedom to choose the scale or units of a coordinate system.

Exercises 2-1

1. Take a sheet of ordinary lined paper and use a lateral edge to make a "ruler" by assigning coordinates to the ends of the lines. Use this ruler to "measure" Figure 2-1. Following the outline of the discussion in Example 1, compare your measurements to each other and to the measurements in Example 1. Find the constants of proportionality which relate the units of your ruler to inches and centimeters.

2. In Example 1 it was asserted that our results agreed within the limitations of accuracy which might be expected. Show that the accuracy of our results is consistent with the accuracy of our measurements.

   We obtained 2.53 rather than 2.54 as the constant of proportionality relating one measurement in centimeters to the corresponding measurement in inches. Justify that this discrepancy is not significant.

3. Assume that the earth is a sphere of radius 3963 miles. A man of extraordinary powers is able to walk completely around the earth at the equator. During this trip his head is always 6 feet farther from the center of the earth than his feet are. Thus the path of the man's head is longer than the path of his feet. Determine how much longer. Let $\pi = 3.1416$. Try to anticipate the appropriate units for the answer.

4. What is the scale of the map on which the "distance" from New York to San Francisco is shown by a line $7\frac{1}{2}$ inches long?

5. (See Exercises 3 and 4.) A model of the earth, or globe, has a 24 inch diameter. What is the scale of this model? How long on the surface of this model would be the "line" from New York to San Francisco?

6. A bicyclist starts along the road at the rate of 8 miles per hour. Two hours later his friend starts after him on a scooter at the rate of 32 kilometers per hour.

   (a) How far apart are the friends one hour later?

   (b) How long and how far have they traveled when they meet?

7. Two bicyclists start at the same time from points 30 miles apart and ride directly toward each other until they meet. The first rides at 4 miles per hour, the second at 5 miles per hour. At the instant they start a preposterous bee starts from the first bicycle toward the second, flying at an unvarying rate of 10 miles per hour. As soon as he meets the second bicycle, the bee turns back and flies to the first, then back to the second, ... He continues to do so until the two riders meet.

   (a) How long in time and distance was the first leg of the bee's flight?
   (b) What was the total length of the bee's flight in time and distance?

2-2. Analytic Representations of Points and Subsets of a Line.

In this section we confine our attention to a line on which a coordinate system has been chosen. We shall let "a" stand for the coordinate of the point A , "b" for that of B , and so forth.

We shall show that the description of betweenness of points is preserved in any linear coordinate system. We shall also show that conditions on points and subsets of a line may be represented by means of relations involving coordinates.

In the SMSG *Geometry* we defined the concept of order for three distinct collinear points. The point B is between the points A and C if and only if $d(A,B) + d(B,C) = d(A,C)$. We proved that when B is between A and C , either $a < b < c$ or $a > b > c$; that is, the coordinate of B is between the coordinates of A and C . We also realized that the converse of this theorem is true. Lastly, we used coordinates to deduce that of three distinct collinear points one and only one is between the other two.

If we change to a coordinate system with a different unit, the measures of distance will change, but the Linear Coordinate System Principle assures us that the corresponding new distances will be proportional to the old. If a, b, and c are the original coordinates of three distinct collinear points and a', b', and c' are new coordinates, then

$$\frac{|a' - b'|}{|a - b|} = \frac{|b' - c'|}{|b - c|} = \frac{|a' - c'|}{|a - c|} .$$

If we let the positive real number k represent the equal ratios above, we may write

$$(1) \qquad |a' - b'| = k|a - b| \ , \ |b' - c'| = k|b - c| \ , \text{ and } \ |a' - c'| = k|a - c| .$$

In the original coordinate system we denote the measures of distance between points by $d(A,B)$, $d(B,C)$, and $d(A,C)$; in the new coordinate system we denote the measures by $d'(A,B)$, $d'(B,C)$, and $d'(A,C)$. By definition,

$$(2) \qquad d(A,B) = |a - b| \ , \ d(B,C) = |b - c| \ , \ d(A,C) = |a - c| \ ,$$

and

$$(3) \quad d'(A,B) = |a' - b'| \ , \ d'(B,C) = |b' - c'| \ , \ d'(A,C) = |a' - c'| \ .$$

Now if B is between A and C , then by definition,

$$d(A,B) + d(B,C) = d(A,C) \ .$$

If we substitute the equal quantities from (2), we obtain

$$|a - b| + |b - c| = |a - c| \ ,$$

which, since $k \neq 0$, is equivalent to

$$k|a - b| + k|b - c| = k|a - c| \ .$$

If we substitute the equal quantities from (1) and (3), we obtain first

$$|a' - b'| + |b' - c'| = |a' - c'|$$

and then

$$d'(A,B) + d'(B,C) = d'(A,C) \ .$$

Thus, the condition describing the order of points on a line is independent of the choice of coordinate system for the line.

Once we have estallished a criterion for describing the order of points on a line, we are able to define such basic geometric entities as segments and rays. We recall that the segment $\overline{PQ}$ is the set which contains P, Q, and all points between P and Q , while the ray $\overrightarrow{PQ}$ is the union of $\overline{PQ}$ and the set of all points R such that Q is between P and R .

We described the points between P and Q as interior points of the segment $\overline{PQ}$. Since an interior point of a segment divides the segment into two other segments, we sometimes call it an *internal point of division*. We identify a point of division of a segment by stating the ratio of the lengths of the new segments.

DEFINITION. A point of division X is said to divide the segment $\overline{PQ}$ in the ratio $\frac{c}{d}$ if and only if

$$\frac{d(P,X)}{d(X,Q)} = \frac{c}{d} .$$

If we let p, q, and x represent the coordinates of P, Q, and X in a coordinate system for the line, we may write

$$\frac{|p - x|}{|x - q|} = \frac{c}{d} .$$

Since X is between P and Q , we know that either $p < x < q$ or $p > x > q$. Thus we may remove the absolute value signs to write either

$$\frac{x - p}{q - x} = \frac{c}{d} \quad \text{or} \quad \frac{p - x}{x - q} = \frac{c}{d} ,$$

which implies

$$dx - dp = cq - cx \quad \text{or} \quad dp - dx = cx - cq .$$

These are both equivalent to

$$cx + dx = dp + cq ,$$

(4) $$x = \frac{dp + cq}{c + d} ,$$

or

(5) $$x = \frac{d}{c + d} p + \frac{c}{c + d} q .$$

Since c and d are either both positive or both negative, x is always defined in terms of p, q, c, and d .

Equation (4) suggests the description of the coordinate of the point of division as a "weighted average" of the coordinates of the endpoints of the segment. The phrase "weighted average" is suggested by the placement of a fulcrum. When two different weights at the ends of a lever are in balance, the fulcrum is closer to the heavier weight than to the lighter weight. In determining a point of division the heavier "weight" is assigned to the coordinate of the closer point and the lighter "weight" to the coordinate of the more remote point.

Example 1. Express the coordinate of the midpoint of segment $\overline{PQ}$ in terms of p and q , the coordinates of the endpoints.

Solution. By definition the midpoint X of a segment is an interior point equidistant from the endpoints. Thus it is a point of division which divides the segment in the ratio one to one. In this case c and d may both be one, and we may write

$$x = \frac{p + q}{2}$$

or

$$x = \frac{1}{2}p + \frac{1}{2}q \, .$$

In Equation (5) above the coefficients of p and q add up to one. If we let $\frac{d}{c + d} = a$ and $\frac{c}{c + d} = b$, we may write

$$x = ap + bq \text{ , where } a > 0 \text{ , } b > 0 \text{ , and } a + b = 1 \, .$$

It is interesting to see what happens here if we omit the requirement that both a and b be positive. Our equation is now

(6) $$x = ap + bq \text{ , where } a + b = 1 \, .$$

If b is zero, a is one and Equation (6) gives the coordinate of P . If a is zero, b is one and Equation (6) gives the coordinate of Q .

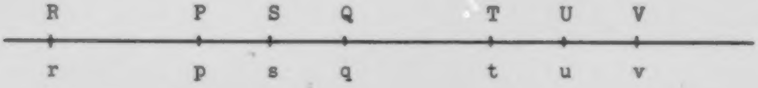


Figure 2-2

In Figure 2-2 we have indicated several points on line $\overleftrightarrow{PQ}$, as well as their coordinates. For convenience let us assume that $r < p < s < q < t < u < v$. We have already seen that if S is the midpoint of $\overline{PQ}$, $s = \frac{1}{2}p + \frac{1}{2}q$; that is, in Equation (6) $a = b = \frac{1}{2}$. Also, p and q are determined by the conditions $a = 1$, $b = 0$ and $a = 0$, $b = 1$ respectively. Let us suppose that $d(P,Q) = d(R,P) = d(Q,T) = d(T,V)$ and that U is the midpoint of $\overline{TV}$. We may determine the coordinates $r, t, u,$ and v in terms of p and q .

The assumption for order of the coordinates permits us to remove the absolute value signs and write:

$$\frac{p - r}{q - r} = \frac{1}{2} , \quad \frac{t - q}{t - p} = \frac{1}{2} , \quad \text{and} \quad \frac{v - q}{v - p} = \frac{2}{3}$$

which imply

$$r = 2p - q , \quad t = -p + 2q , \quad \text{and} \quad v = -2p + 3q \quad \text{respectively.}$$

Since U is the midpoint of $\overline{TV}$,

$$\begin{aligned} u &= \frac{1}{2}t + \frac{1}{2}v \\ &= \frac{1}{2}(-p + 2q) + \frac{1}{2}(-2p + 3q) \\ &= -\frac{3}{2}p + \frac{5}{2}q . \end{aligned}$$

Had we chosen to orient the coordinate system in the opposite direction, we would have obtained the same results.

In every case above the sum of the coefficients of p and q is one. This suggests that any point on the line may be represented by adopting appropriate coefficients in Equation (6). This is true, although we do not prove it here. When a variable is expressed by a form similar to the right side of Equation (6), we say that it is expressed as a linear combination of p and q . We shall have occasion to develop this idea in the next chapter. We may describe our conjecture here by saying that the coordinate of any point on a line may be expressed as a linear combination of the coordinates of two given distinct points on the line.

In view of the restriction on Equation (6), we really need only one variable to represent the coefficients. If we let $t = a$, then $b = 1 - t$ and we may write

(7) $$x = tp + (1 - t)q \quad \text{where } t \text{ is any real number.}$$

Thus the variable x is related to the constants p and q by a second variable t . It is clear what x represents; it is the coordinate of a point on the line. We know that t represents a real number and we can see that each value of t determines a unique value of x , but it is not immediately clear what t names or measures. Our primary interest is in the variable x ; our interest in t is definitely subordinate. When we express one or more variables in terms of yet another variable, we frequently say that we have a parametric representation. The other variable is called a parameter. We shall want to develop this idea in Chapter 5.

In the present case we see that when $t = 0$, $x = q$; when $t = 1$, $x = p$; and when $t = \frac{1}{2}$, $x = \frac{1}{2}p + \frac{1}{2}q$. This suggests the explanation of the role of t . The Linear Coordinate System Principle assures us that there exists another coordinate system on the line $\overleftrightarrow{PQ}$ in which the coordinate of Q is zero and the coordinate of P is one. A point whose coordinate is represented by t in the latter coordinate system is represented by x in the former coordinate system. The coordinates in the two coordinate systems are related by Equation (7).

We have developed several different ways of describing a point on a line by means of equations involving coordinates. We call such descriptions analytic representations. We now turn to analytic representations of subsets of the line.

In earlier courses you have studied a number of subsets of a line. Among them are the following:

$\overleftrightarrow{AB}$, the line through A and B ;

$\overrightarrow{AB}$, the ray whose endpoint is A and which contains B ;

$\overline{AB}$, the segment with endpoints A and B .

It is possible to represent these and many other subsets of a line analytically. We consider a number of examples below, and ask you to study others in the exercises. In what follows, when we say that b is between a and c (a , b, and c real numbers), we mean that either $a < b < c$ or $c < b < a$. Then B is between A and C if and only if b is between a and c .

$\overleftrightarrow{AB}$ consists of all points X with any real coordinate x . We can say this in the form

$$\overleftrightarrow{AB} = \{X:\ x \text{ is real}\}$$

or in the form

$$\overleftrightarrow{AB} = \{X:\ x^2 \geq 0\} .$$

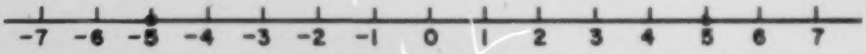
Further

$$\overline{AB} = \{X:\ a \leq x \leq b \text{ or } b \leq x \leq a\}$$

$$\overrightarrow{AB} = \{X:\ b > a \text{ and } x \geq a \text{, or } b < a \text{ and } x \leq a\} .$$

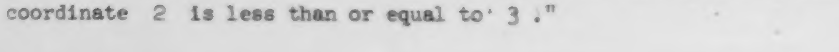
There are two related problems which crop up frequently in analytic geometry, one of which is illustrated above. A set S of points may be specified by geometric conditions, and we may ask for an analytic condition satisfied by the coordinates of points of S but not by those of any other points. On the other hand, we may be given an analytic condition and want to know what points have coordinates satisfying it. You have met both these problems before. The analytic condition was usually an equation, but you have also considered inequalities, and some of the conditions considered below involve other relations. When a set of points consists of those points whose coordinates satisfy a certain condition, we call the set the *graph* (or *locus*) of the condition; we call the condition a *condition for* (or of) *the set*. These ideas prove more interesting and more important in a plane and in space, but we shall discuss some examples on a line and ask you to work on others.

Example 1. The graph of $|x| = 5$, which is also the graph of $x^2 = 25$, is the set of points with coordinates ± 5.



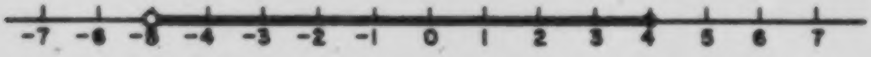
This illustrates the fact that there may be different conditions for the same set of points. (Of course this raises the question of whether the conditions are really different, but at least they were expressed differently.)

Example 2. To find the graph of $|3x - 6| \le 9$, we observe that $|3x - 6| \le 9$ is equivalent to $3|x - 2| \le 9$, or $|x - 2| \le 3$. The graph is shown below.



The use of the absolute value in measuring distance is an aid in finding the graph. Thus, the graph of the solution set of $|x - 2| \le 3$ may be interpreted as "the set of all points of the line whose distance from the point with coordinate 2 is less than or equal to 3."

Example 3. Find an analytic condition for the set of points shown below.



(The heavy dot is a device for indicating that the right endpoint is in the set.) An analytic condition for this set is

$$-5 < x \leq 4 \; .$$

Example 4. Let the coordinates of points O , A , X , be 0 , a , x , respectively. Find all points X such that $2d(O,X) + 3d(X,A) = d(O,A)$.

Solution. For any X , $d(O,X) + d(X,A) \geq d(O,A)$. Then, unless $d(O,X) = d(X,A) = 0$, we have

$$2d(O,X) + 3d(X,A) > d(O,A) \; .$$

Thus there is no solution unless $O = X = A$.

Exercises 2-2

1. Represent graphically:

(a) $r^2 = 4$

(b) $(x - 3)^2 = 4$

(c) $|r - 3| = 2$

(d) $x + 3 < 7$

(e) $5 \leq 2 - x$

(f) $|t + 3| < 3$

(g) $x(x - 1) > 0$

(h) $(x - 1)(x + 2) \leq 0$

(i) $x^2 + 4 < - 4x$

(j) $|2y - 4| = 6$

(k) $|x - 1.123| < .456$

(l) $|2s + 2| < 4$

(m) $|3x + 2| = 1$

(n) $\sin x\pi = 0$

(o) $2 \sin x\pi = 1$

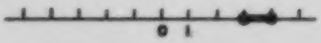
(p) $\cos \theta > 0$

(q) $|x - a| < \delta$, where $a = 2.35$ and $\delta = 0.44$

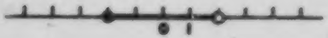
(r) $|x - a| < \delta$, where $a = 0.44$ and $\delta = 2.35$

2. Represent analytically:

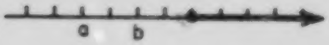
(a)



(b)



(c)



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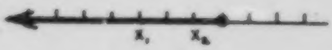
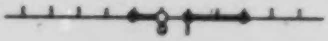
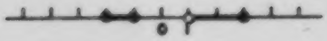


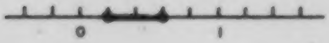
(e)



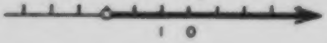
(f)



(g)

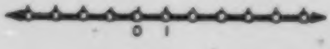


(h)

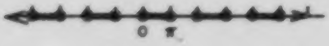


(For Parts (i) and (j) assume the same pattern throughout the line.)

(i)



(j)



3. Points O , U , A , and X have coordinates 0 , 1 , a , and x respectively. Find all values of x that satisfy each of the following conditions:

(a) $d(O,X) = 3d(O,A)$

(b) $d(O,X) + d(U,X) = d(O,U)$.

4. If P and Q have the coordinates given, and if M , A , and B are the midpoint and the two trisection points of $\overline{PQ}$ respectively, find, in each case, the coordinates m , a , and b :

(a) $p = 3$, $q = 12$

(b) $p = -2$, $q = 13$

(c) $p = r + s$, $q = r - s$

(d) $p = (r + t) - 2$, $q = (r + t) + 4$

(e) $p = 2r$, $q = 3t$

(f) $p = 2r + 3s$, $q = 3r - 2s$

(g) $p = r^2 - r$, $q = s^2 - s$

(h) $p = r$, $q = s$.

5. In the equation of the line $\overleftrightarrow{PQ}$

$$x = ap + bq \text{ , where } a + b = 1 \text{ ,}$$

x , p , and q are the coordinates of the points X , P , and Q respectively.

Find the relative positions of X , P , and Q if

(a) $a = 0$	(d) $a < 0$
(b) $a = 1$	(e) $a > 1$
(c) $0 < a < 1$	(f) $b > 1$.

6. In the equation of the line $\overleftrightarrow{PQ}$

$$x = tp + (1 - t)q \text{ , where } t \text{ is real,}$$

x , p , and q are the coordinates of the points X , P , and Q respectively. For what value(s) of t is

(a) $d(P,X) = 2d(Q,X)$	(c) $d(X,P) = 2d(P,Q)$
(b) $2d(P,X) = d(Q,X)$	(d) $d(P,Q) = d(Q,X)$

Exercises 7-10 are based upon the following situation:

Points A , B , C , D , and E are on the edge of an ordinary 12 inch ruler at positions corresponding to 1 , $1\frac{1}{2}$, $2\frac{1}{2}$, $4\frac{1}{2}$, and 9 respectively. These numbers are the inch-coordinates a , b , c , d , and e , of the corresponding points.

7. Find the ratios (a) $\frac{d(A,B)}{d(B,C)}$, (b) $\frac{d(B,C)}{d(C,D)}$, and (c) $\frac{d(C,D)}{d(D,E)}$.

8. Express

   (a) b as a linear combination of a and c .
   (b) c as a linear combination of b and d .
   (c) d as a linear combination of c and e .

9. Find the inch-coordinates of the trisection points of $\overline{AC}$; of $\overline{BD}$; of $\overline{CE}$.

10. Find the inch-coordinates of points P , Q , and R such that

$$\frac{d(A,B)}{d(B,P)} = \frac{2}{3} \text{ , } \frac{d(B,C)}{d(C,Q)} = \frac{2}{3} \text{ , and } \frac{d(C,D)}{d(E,R)} = \frac{2}{3} \text{ .}$$

2-3. Coordinates in a Plane.

You will recall that the points of a plane can be put into one-to-one correspondence with the ordered pairs of real numbers in the following way. Any two perpendicular lines in the plane are selected as reference lines or axes. They are called the x-axis and the y-axis. The intersection of these lines is called the origin and denoted by O . On each axis we use a coordinate system with O as origin. Normally the two coordinate systems should use the same units. It is possible to use different coordinate systems on the two axes, but this introduces complications, a few of which will be considered in exercises. If P is any point in the plane, let a and b be the coordinates of the projections of P onto the x-axis and y-axis respectively. Then to P we assign the ordered pair (a,b) of real numbers (rectangular coordinates). The first is called the x-coordinate or abscissa of P , the second the y-coordinate or ordinate of P . Conversely, if (a,b) is an ordered pair of real numbers, there is a unique point P with abscissa a and ordinate b . It is the intersection of the line perpendicular to the x-axis at the point on that axis with coordinate a , and the line perpendicular to the y-axis at the point on that axis with coordinate b .

In sketches it is customary, though not necessary, to show the x-axis horizontal with its positive half to the right, the y-axis vertical with its positive half upward. In all sketches we place an x by the end of the line representing the positive half of the x-axis and a y by the end of the line representing the positive half of the y-axis. This is essential when we do not indicate the coordinates of any points on the axes.

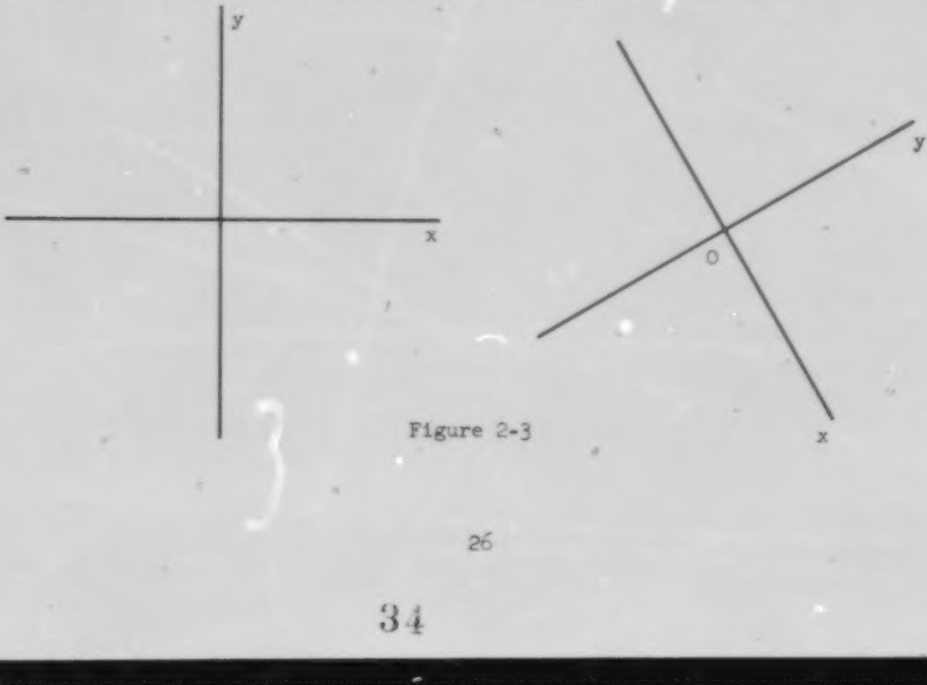


Figure 2-3

We customarily reserve the letter 0 to represent the origin, but do not always include it on a sketch unless we refer to it.

You will also recall that if $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$, then the distance between the two points is

$$d(P_0,P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} .$$

We turn now to the problem of expressing the coordinates of any point $P = (x,y)$ of the line L determined by the distinct points $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$ in terms of the coordinates of P_0 and P_1. Let us assume for the time being that $x_0 \neq x_1$ and $y_0 \neq y_1$.

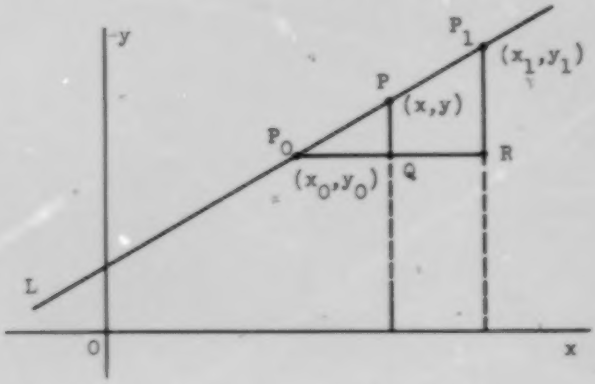


Figure 2-4

In Figure 2-4 $\overleftrightarrow{P_0Q}$ is perpendicular to the y-axis, $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P_1R}$ to the x-axis. Then triangles P_0QP and P_0RP_1 are similar, and hence

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} . \tag{1}$$

Be sure that you see that the same equation holds if the order of P_0, P_1, and P is different.

If the point P is an internal point of division which divides the segment $\overline{P_0P_1}$ in the ratio $\frac{c}{d}$, then each member of Equation (1) is equal to $\frac{c}{c + d}$ and we may write

$$\frac{x - x_0}{x_1 - x_0} = \frac{c}{c + d} \quad \text{and} \quad \frac{y - y_0}{y_1 - y_0} = \frac{c}{c + d} .$$

If we solve these equations for x and y, we obtain

$$x = \frac{dx_0 + cx_1}{c + d} \quad \text{and} \quad y = \frac{dy_0 + cy_1}{c + d}, \tag{2}$$

in which the coordinates of the point of division are expressed as weighted averages of the coordinates of the endpoints of the segment.

We are now in a position to follow exactly the same development as in Section 2-2.

If P is the midpoint of $\overline{P_0P_1}$, it divides the segment in the ratio one to one. In this case we may let $c = d = 1$ and write

$$x = \frac{x_0 + x_1}{2} \quad \text{and} \quad y = \frac{y_0 + y_1}{2}.$$

If in Equations (2) we let $a = \frac{d}{c + d}$ and $b = \frac{c}{c + d}$, we may write

$x = ax_0 + bx_1$ and $y = ay_0 + by_1$, where $a > 0$, $b > 0$, and $a + b = 1$.

If we omit the requirement that a and b be positive, we obtain

$$x = ax_0 + bx_1 \quad \text{and} \quad y = ay_0 + by_1, \text{ where } a + b = 1. \tag{3}$$

An analysis similar to that of Equation (6) in the previous section would suggest that each point $P = (x,y)$ on $\overleftrightarrow{P_0P_1}$ corresponds to a unique choice of numbers for a and b in Equations (3), and conversely each pair (a,b) in Equations (3) corresponds to a unique point on $\overleftrightarrow{P_0P_1}$. Thus the x-coordinate of a point on a line may be represented by a linear combination of the x-coordinates of two given distinct points on the line, while the y-coordinate is represented by the same linear combination of the y-coordinates of the given points.

Lastly, we recognize that, because of the restriction on the coefficients in Equations (3), one variable will suffice. If we let $t = b$, then $a = 1 - t$ and we obtain

$$x = (1 - t)x_0 + tx_1 \quad \text{and} \quad y = (1 - t)y_0 + ty_1$$

or

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \end{aligned} \quad \text{where } t \text{ is real.} \tag{4}$$

This is a parametric representation of the point $P = (x,y)$ on the line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. As we shall see in Chapter 5, this representation is not only useful; for certain problems it is essential. As we observed in the previous section, the parameter t represents the coordinate of P in the linear coordinate system with origin P_0 and unit-point P_1.

The coordinate system for a plane which we have described and used above is called a rectangular or Cartesian coordinate system. The name "Cartesian" comes from Descartes, who is credited with being the first to introduce the theory of algebra into the study of geometry.

Exercises 2-3

1. If P and Q have the coordinates given, and if M, A, and B are the midpoint and the two trisection points of $\overline{PQ}$ respectively, find the coordinates of M, A, and B in each case:

   (a) $P = (0,0)$, $Q = (6,9)$
   (b) $P = (2,3)$, $Q = (8,12)$
   (c) $P = (5,12)$, $Q = (6,-7)$
   (d) $P = (4,-3)$, $Q = (-9,10)$
   (e) $P = (-6,-3)$, $Q = (6,3)$
   (f) $P = (-3,-6)$, $Q = (-6,-3)$
   (g) $P = (p_1,p_2)$, $Q = (q_1,q_2)$
   (h) $P = (2s,5t)$, $Q = (s,-2t)$
   (i) $P = (4r + 2s, -3r + s)$, $Q = (-r - s, -r - 2s)$

2. Let $P = (x,y)$ be a point on line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. Express x as a linear combination of x_0 and x_1 and y as the same linear combination of y_0 and y_1 in the following cases:

   (a) $P_0 = (2,3)$, $P_1 = (6,1)$
   (b) $P_0 = (-4,5)$, $P_1 = (2,-7)$
   (c) $P_0 = (-3,-6)$, $P_1 = (-6,4)$

3. Let $P = (x,y)$ be a point on line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. In the following cases express coordinates of P by a parametric representation. Choose the parameter t so that $(x,y) = P_0$ when $t = 0$ and $(x,y) = P_1$ when $t = 1$.

   (a) $P_0 = (2,3)$, $P_1 = (6,1)$

   (b) $P_0 = (-4,5)$, $P_1 = (2,-7)$

   (c) $P_0 = (-3,-6)$, $P_1 = (-6,4)$

4. In the development of Equation (1) in Section 2-3, we assumed that $x_0 \neq x_1$ and $y_0 \neq y_1$. If $x_0 = x_1$ or $y_0 = y_1$, this equation does not hold, but Equation (2) in Section 2-3 does apply. Consequently, the rest of the development is valid in either of these cases. Justify that Equation (2) applies when the conditions are relaxed. [Hint: Show that the problem reduces to the situation discussed in Section 2-2.]

5. Apply the condition given by Equation (1) to decide whether the points A , B , and C with the coordinates given, are collinear. How can you use the formula for the distance between two points to determine whether three points are collinear? Use this method to check your answers.

   (a) $A = (7,0)$, $B = (-3,-6)$, $C = (22,9)$

   (b) $A = (-1,4)$, $B = (3,-14)$, $C = (-5,-6)$

6. For what value of h is the point $P = (h,-3)$ on the line determined by $A = (1,-1)$ and $B = (4,7)$?

2-4. Polar Coordinates.

A rectangular coordinate system is certainly the most frequently employed coordinate system, but it is not always the best choice for a given problem.

The rectangular coordinate system is based upon a grid composed of two mutually perpendicular systems of evenly spaced parallel lines in a plane. An alternative is the polar coordinate system, which is based upon a grid composed of a system of concentric circles and a system of rays emanating from the common center of the circles.

The paths from one point to another on a rectangular grid usually entail motion along two adjacent sides of a rectangle, but the natural paths of physical objects are usually more direct. A football player does not pass the ball to follow the deceptive path of a receiver. Rather he looks for the receiver in a certain area. If he finds the receiver uncovered, he will try to pass the ball just so far in the direction of the receiver. To apply this idea in the plane we require a frame of reference. The frame of reference consists of a fixed point O, called the *pole*, and a fixed ray $\overrightarrow{OM}$, called the *polar axis*. The ray has the non-negative part of a linear coordinate system with the origin at O. The position of a point P is uniquely determined by r and θ, its *polar coordinates* (Figure 2-5a).

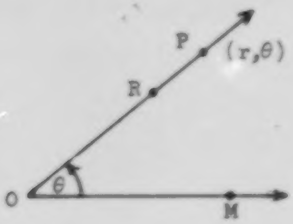


Figure 2-5a

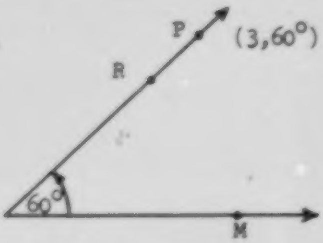


Figure 2-5b

The *polar angle* θ is an angle generated by rotating a ray $\overrightarrow{OR}$ about O from $\overrightarrow{OM}$ in either direction as far as desired and terminating the rotation in a position such that the line $\overrightarrow{OR}$ contains P. If we rotate $\overrightarrow{OR}$ in a counterclockwise direction, $\angle\theta$ has a positive measure; if $\overrightarrow{OR}$ is rotated clockwise, then $\angle\theta$ has a negative measure.

DEFINITION. If $\overrightarrow{OR}$ contains P, then the *polar distance* $r = d(O,P)$; if P lies on the ray opposite to $\overrightarrow{OR}$, then $r = -d(O,P)$.

Commonly used units of measure for polar angles are degrees and radians. When the usual symbols for numerical measure of angles in degrees, minutes and seconds are omitted, it is understood that radian measure is intended.

The polar coordinates of a point are written as an ordered pair (r,θ), where r is the polar distance and θ is a measure of the polar angle. If the angle is measured in degrees, the symbolism alone indicates that the ordered pair represents polar coordinates. If the measure of the angle is given in radians, the ordered pair of real numbers is indistinguishable from the notation used in rectangular coordinates. If the context does not make clear that these are polar coordinates, we must say so explicitly. If no indication is given, we shall assume that rectangular coordinates are intended.

The pole is a special point. When $r = 0$, the pole is described. In this case $\angle\theta$ may have any measure. $(0,0)$, $(0,60^\circ)$, $(0,180^\circ)$, and $(0,\frac{\pi}{2})$ are all names for the pole. We usually write $(0,\theta)$ to indicate that θ may be any number. The pole is not the only point whose representation is not unique.

A rectangular coordinate system establishes a one-to-one correspondence between points in a plane and ordered pairs of real numbers. It is important to observe that a polar coordinate system does not. In polar coordinates each ordered pair corresponds to a unique point in the plane, but each point is represented by infinitely many ordered pairs of numbers.

For example, some other coordinates for the point P shown in Figure 2-5b are $(3,420^\circ)$, $(3,-300^\circ)$, and $(-3,-\frac{2}{3}\pi)$. See Figure 2-6.

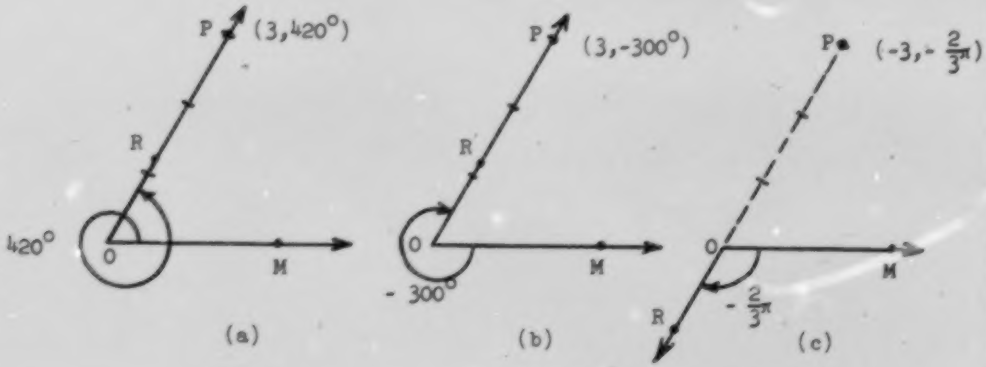


Figure 2-6

In subsequent figures we shall delete the arrowhead from all rays except the polar axis.

The lack of a one-to-one correspondence between points and ordered pairs of numbers necessitates care when we use polar coordinates, but the advantages are sometimes great indeed. For example, the figures which we have used here may remind you of the figures which illustrated the definitions of the trigonometric or circular functions. As you will discover in subsequent chapters, the analytic representations of these functions and allied relations are often simpler in polar coordinates.

Example 1. Plot the points A , B , C , and D , which have polar coordinates $(2,45^\circ)$, $(3,-120^\circ)$, $(1,\frac{\pi}{3})$, and $(-2,-\frac{\pi}{4})$ respectively.

Solution.

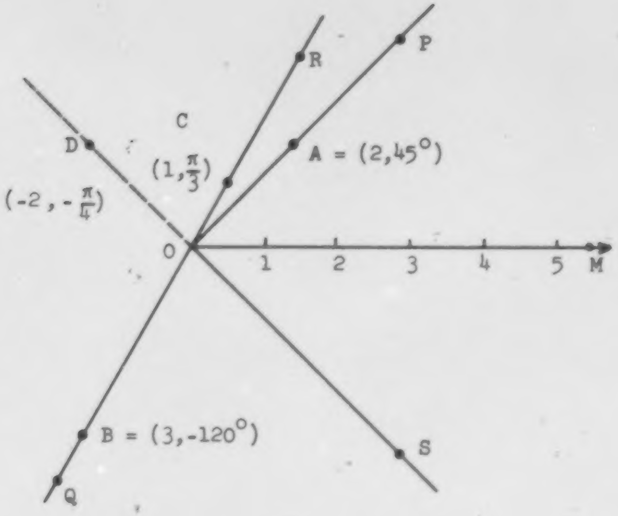


Figure 2-7

Since a measure of $\angle POM = 45^\circ$, A is the point on $\overrightarrow{OP}$ such that $d(O,A) = 2$. A measure of $\angle QOM = -120^\circ$ and B is the point on $\overrightarrow{OQ}$ such that $d(O,B) = 3$. A measure of $\angle ROM = \frac{\pi}{3}$ and C is the point on $\overrightarrow{OR}$ such that $d(O,C) = 1$. Lastly, a measure of $\angle SOM = -\frac{\pi}{4}$, but since the polar distance is negative, D is the point on the ray opposite to $\overrightarrow{OS}$ such that $d(O,D) = 2$.

Example 2. Find four pairs of polar coordinates, two in degrees and two in radians, for each of the points A , B , and C in Figure 2-8.

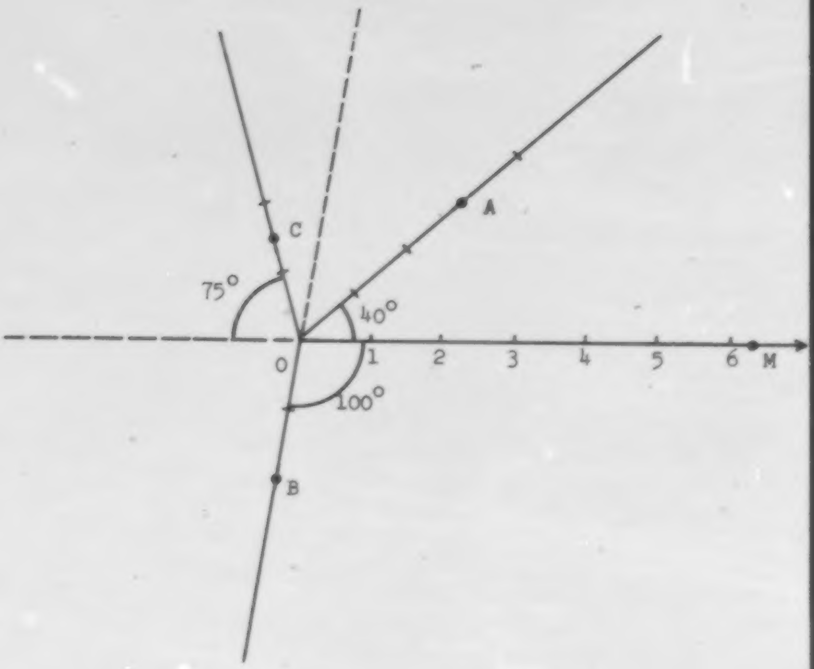


Figure 2-8

Solution. A simple representation for A is $(3,40^\circ)$, but we may also use $(3,-320^\circ)$, $(3,\frac{2\pi}{9})$, and $(-3,\frac{11\pi}{9})$. (There are others, of course.) $B = (2,-100^\circ)$, $(-2,80^\circ)$, $(2,\frac{-5\pi}{9})$, and $(-2,\frac{4\pi}{9})$. $C = (1\frac{1}{2},105^\circ)$, $(1\frac{1}{2},465^\circ)$, $(1\frac{1}{2},\frac{7\pi}{12})$, and $(-1\frac{1}{2},\frac{19\pi}{12})$.

We mentioned that any pair of perpendicular lines in a plane may be chosen as the reference axes for a rectangular coordinate system. Any ray in a plane may be chosen for the polar axis in introducing a polar coordinate system. When we are solving a problem using coordinates, this freedom enables us to choose a system which will simplify the computation. Because we wish to keep this in mind, we state the following principle:

COORDINATE PLANE PRINCIPLE. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two perpendicular lines intersecting at O ($O \neq A$ and $O \neq C$), there exists a rectangular coordinate system in the plane of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ such that

(i) $\overleftrightarrow{AB}$ is the x-axis, $\overleftrightarrow{CD}$ is the y-axis

and

(ii) in the coordinate systems on the axes, the coordinates of A and C are positive.

In any plane containing the ray $\overrightarrow{OM}$ there exists a polar coordinate system such that $\overrightarrow{OM}$ is the polar axis.

In some situations we must use both rectangular and polar coordinate systems in the same plane. Usually we let the polar axis coincide with the non-negative half of the x-axis. Coordinates in both systems are assigned to each point in the plane, but we shall need equations relating the coordinates in order to change back and forth.

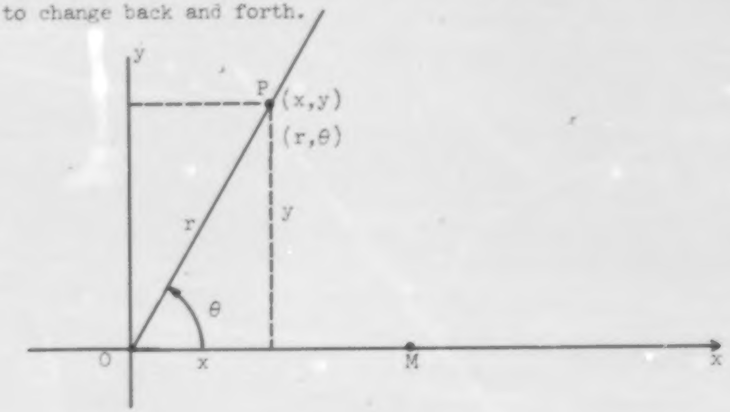


Figure 2-9

In Figure 2-9, we see that

(1)
$$x = r \cos \theta$$
$$y = r \sin \theta$$

and

(2)
$$r^2 = x^2 + y^2$$
$$\tan \theta = \frac{y}{x}, \text{ where } x \neq 0 .$$

In Equations (2) we note that, as we might have expected, r and θ are not uniquely defined. You should verify these equations for points in other quadrants.

We may use Equations (1) to transform from polar coordinates to rectangular coordinates and Equations (2) to find polar coordinates for points whose rectangular coordinates are known.

Example 3. Find the rectangular coordinates of the point designated in polar form by $(8,-60^\circ)$.

Solution.

$$x = 8\cos(-60^\circ) = 8(\tfrac{1}{2}) = 4$$

$$y = 8\sin(-60^\circ) = 8(-\tfrac{1}{2}\sqrt{3}) = -4\sqrt{3}$$

Example 4. Find a polar representation for the point with rectangular form $P = (-2,-2)$.

Solution. $r^2 = (-2)^2 + (-2)^2 = 8$; therefore, $r = \pm 2\sqrt{2}$. Also, $\tan\theta = \frac{-2}{-2} = 1$; hence, $\theta = \frac{\pi}{4} + n\pi$, n an integer . It is necessary to match the values of r and θ which correctly locate P . For example, $(2\sqrt{2}, \frac{\pi}{4})$ is not a correct solution, as these coordinates locate a point in the first quadrant. But $(2\sqrt{2}, \frac{5\pi}{4})$ and $(-2\sqrt{2}, \frac{\pi}{4})$ are two of the possible correct designations for P .

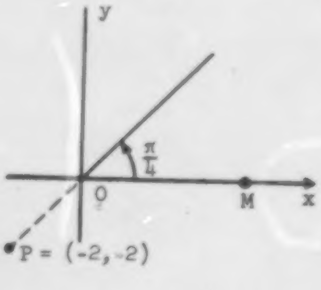


Figure 2-10

Example 5. Find the distance between the points P_1 and P_2 whose polar coordinates are (r_1,θ_1) and (r_2,θ_2) respectively.

Solution. We have an expression for the distance between two points in terms of their rectangular coordinates,

$$(3) \qquad d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ .$$

We may use this expression if we transform the coordinates of P_1 and P_2 to rectangular form. We use Equations (1) to obtain

$$x_1 = r_1 \cos \theta_1 \ , \ y_1 = r_1 \sin \theta_1$$

$$x_2 = r_2 \cos \theta_2 \ , \ y_2 = r_2 \sin \theta_2 \ .$$

We square both members of Equation (3) and substitute these values to obtain

$$\left(d(P_1,P_2)\right)^2 = (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2$$

or

$$\left(d(P_1,P_2)\right)^2 = r_2^2 \cos^2 \theta_2 - 2r_1r_2 \cos \theta_2 \cos \theta_1 + r_1^2 \cos^2 \theta_1$$
$$+ r_2^2 \sin^2 \theta_2 - 2r_1r_2 \sin \theta_2 \sin \theta_1 + r_1^2 \sin^2 \theta_1 \ .$$

If we apply the distributive and other laws, this becomes

$$\left(d(P_1,P_2)\right)^2 = r_1^2(\cos^2 \theta_1 + \sin^2 \theta_1) + r_2^2(\cos^2 \theta_2 + \sin^2 \theta_2)$$
$$- 2r_1r_2(\cos \theta_2 \cos \theta_1 + \sin \theta_2 \sin \theta_1) \ .$$

Now $\qquad \cos^2 \theta_1 + \sin^2 \theta_1 = 1 \ , \ \cos^2 \theta_2 + \sin^2 \theta_2 = 1 \ ,$

and $\qquad \cos \theta_2 \cos \theta_1 + \sin \theta_2 \sin \theta_1 = \cos(\theta_2 - \theta_1) \ .$

We substitute these equivalent values to obtain

$$(3) \qquad \left(d(P_1,P_2)\right)^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1) \ .$$

We might have obtained this expression directly by applying the Law of Cosines to triangle OP_1P_2 in Figure 2-11.

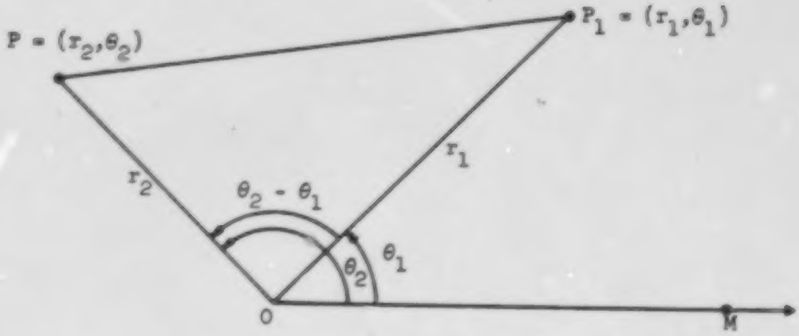


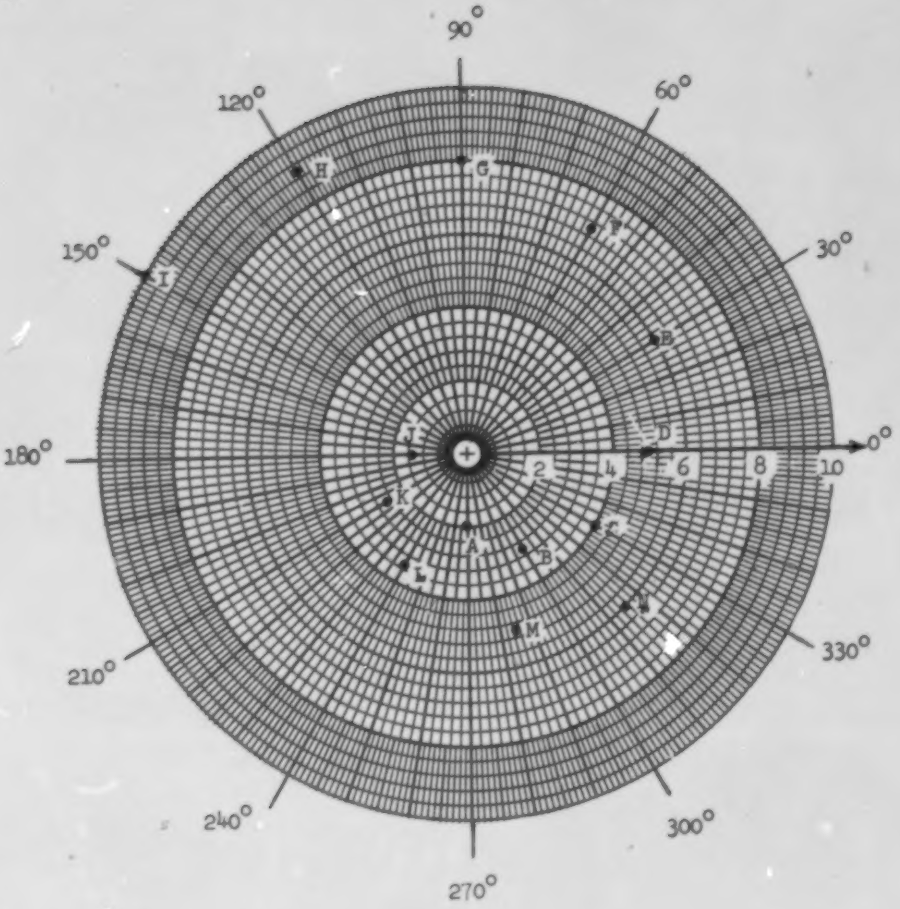
Figure 2-11

Thus the distance formula in polar coordinates is an application of the Law of Cosines.

Exercises 2-4

1. Plot the following points and for each list three pairs of coordinates: $(5,135^\circ)$, $(2,90^\circ)$, $(-4,45^\circ)$, $(3,-120^\circ)$.
2. Plot the points whose polar coordinates are $(-2,45^\circ)$, $(-4,210^\circ)$, $(3,2)$, $(-3,-\frac{3}{2})$, $(4,0^\circ)$, $(0,\frac{\pi}{2})$, $(-4,180^\circ)$.
3. Plot the vertices of an equilateral triangle, the centroid coincident with the pole and a vertex on the polar axis, and give polar coordinates of the vertices.
4. Draw graphs representing the set of points $\{(r,\theta) : r = 4\}$; the set of points $\{(r,\theta) : \theta = 45^\circ\}$.

5.


90°
120°
60°
H
G
F
150°
30°
I
E
J
D
180°
0°
2
4
6
8
10
K
C
A
B
L
N
M
210°
330°
240°
300°
270°

For each of the points indicated on the preceding diagram give five pairs of polar coordinates; in the first pair have $r > 0$, and $0^\circ \le \theta < 360^\circ$, in the second pair have $r > 0$, and $-360^\circ < \theta \le 0^\circ$, in the third pair have $r < 0$, and $0^\circ \le \theta < 360^\circ$, in the fourth pair have $0 \le \theta < \pi$, in the fifth pair have $0^\circ \le \theta < 180^\circ$.

6. Find the rectangular representation of the points whose polar coordinates are

   (a) $(0,90^\circ)$
   (b) $(\sqrt{2},-45^\circ)$
   (c) $(5,420^\circ)$
   (d) $(4,0^\circ)$
   (e) $(1,\pi)$
   (f) $(\sqrt{2},\frac{5\pi}{2})$
   (g) $(-2,\frac{1}{3}\pi)$
   (h) $(2,-\frac{\pi}{4})$

7. Write a polar representation for the points whose rectangular coordinates are

   (a) $(1,1)$
   (b) $(2,-2)$
   (c) $(p,0)$
   (d) $(0,q)$
   (e) $(-\sqrt{3},1)$
   (f) $(-1,-\sqrt{3})$
   (g) $(5,2)$
   (h) $(-4,1)$

8. Use polar coordinates to find the distance between the points A and B. Then change to rectangular coordinates and verify your result.

   (a) $A = (2,150^\circ)$, $B = (4,210^\circ)$
   (b) $A = (5,\frac{5}{4}\pi)$, $B = (12,\frac{7}{4}\pi)$

9. Find the distance between each of the following pairs of points.

   (a) $A = (3,0^\circ)$, $B = (5,90^\circ)$
   (b) $A = (2,37^\circ)$, $B = (3,100^\circ)$
   (c) $A = (6,100^\circ)$, $B = (8,400^\circ)$
   (d) $A = (-1,45^\circ)$, $B = (3,165^\circ)$
   (e) $A = (3,20^\circ)$, $B = (5,140^\circ)$
   (f) $A = (5,-60^\circ)$, $B = (10,-330^\circ)$

10. On a polar graph chart such as in Exercise 5 construct a hexagon with vertices $(10,0^\circ)$, $(10,60^\circ)$, etc. Then construct all its diagonals and write the coordinates of all their intersections (other than the pole).

11. Let (r_0,θ_0) represent a point P. Find general expressions for all the possible polar coordinates of P

   (a) when θ_0 is in degrees and
   (b) when θ_0 is in radians.

2-5. Lines in a Plane.

Now that we have developed coordinate systems for planes, we are able to discuss analytic representations of subsets of planes. We start with the line.

Symmetric Form. In Section 2-3 we developed Equation (1),

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}, \tag{1}$$

which is the analytic condition describing a point $P = (x,y)$ on the oblique line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. (We note that the requirement that the line be oblique ensures that the denominator in each member is not zero.)

Since every point on the line may be described in this way,

$$\left\{(x,y) : \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}\right\} = \overleftrightarrow{P_0P_1} .$$

We call Equation (1) a symmetric form of the equation of a line.

Example 1. A symmetric form of an equation of the line containing the points $(2,3)$ and $(4,-1)$ is

$$\frac{x - 2}{4 - 2} = \frac{y - 3}{-1 - 3} \quad \text{or} \quad \frac{x - 2}{2} = \frac{y - 3}{-4} .$$

Two-Point Form. If we reverse the order of the members of Equation (1) and multiply by $(y_1 - y_0)$, we obtain

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) . \tag{2}$$

We call Equation (2) a two-point form of the equation of a line.

Example 2. A two-point form for an equation of the line containing the points $(1,-2)$ and $(4,5)$ is

$$y + 2 = \frac{5 + 2}{4 - 1}(x - 1) \quad \text{or} \quad y + 2 = \frac{7}{3}(x - 1) .$$

We note that in Equation (2) the quotient of differences, or the difference quotient, $\frac{y_1 - y_0}{x_1 - x_0}$ is, by definition, the slope of the segment $\overline{P_0P_1}$. In your study of geometry you may have used similar triangles to prove that every segment of a given line has the same slope. We define the slope of a line to be the slope of all the segments on that line. We denote the slope of a segment or line by m .

The two-point form is not precisely equivalent to the symmetric form, since it is also defined when $y_0 = y_1$ or $y_1 - y_0 = 0$. In this case the line $\overleftrightarrow{P_0P_1}$ is parallel to the x-axis, has a slope of zero, and is represented by the equation $y - y_0 = 0$.

If $x_0 = x_1$ or $x_1 - x_0 = 0$, neither the symmetric form nor the two-point form as given in Equation (2) is defined. In this case an alternative two-point form

$$(3) \qquad x - x_0 = \frac{x_1 - x_0}{y_1 - y_0}(y - y_0)$$

is defined. In this case the line $\overleftrightarrow{P_0P_1}$ has no slope, is parallel to the y-axis, and is represented by the equation $x - x_0 = 0$.

If $x_0 = x_1$ and $y_0 = y_1$, the points P_0 and P_1 are, of course, not distinct and no line is determined.

Example 3.

(a) The line containing the points (1,2) and (4,3) has slope $m = \frac{3 - 2}{4 - 1} = \frac{1}{3}$ and has as an equation in two-point form

$$y - 2 = \frac{3 - 2}{4 - 1}(x - 4) \quad \text{or} \quad y - 2 = \frac{1}{3}(x - 4) .$$

(b) The line containing the points (2,3) and (4,3) has slope $m = \frac{3 - 3}{4 - 2} = 0$ and has an equation in two-point form

$$y - 3 = \frac{3 - 3}{4 - 2}(x - 2) \quad \text{or} \quad y - 3 = 0 .$$

The line containing the points $(1,3)$ and $(1,5)$ has no slope since $\frac{5-3}{1-1} = \frac{2}{0}$ is not defined. However, it has an equation in an alternative two-point form:

$$x - 1 = \frac{1-1}{5-3}(y - 3) \quad \text{or} \quad x - 1 = 0 \, .$$

Point-Slope Form. Since a line is determined by two distinct points, a line in a plane with a rectangular coordinate system is determined by the coordinates of two points on the line. If a line has slope, it is also determined by its slope and the coordinates of one of its points.

If a line has slope m and contains the point (x_0,y_0), we may replace the difference quotient in Equation (2) by m to obtain

$$y - y_0 = m(x - x_0) \, . \tag{4}$$

We call Equation (4) a point-slope form of the equation of a line.

Example 4. A point-slope form of the line which contains the point $(5,-3)$ and has slope $\frac{2}{3}$ is

$$y + 3 = \frac{2}{3}(x - 5) \, .$$

Inclination. Occasionally we wish to describe a line, not by its slope, but by an angle related to the slope.

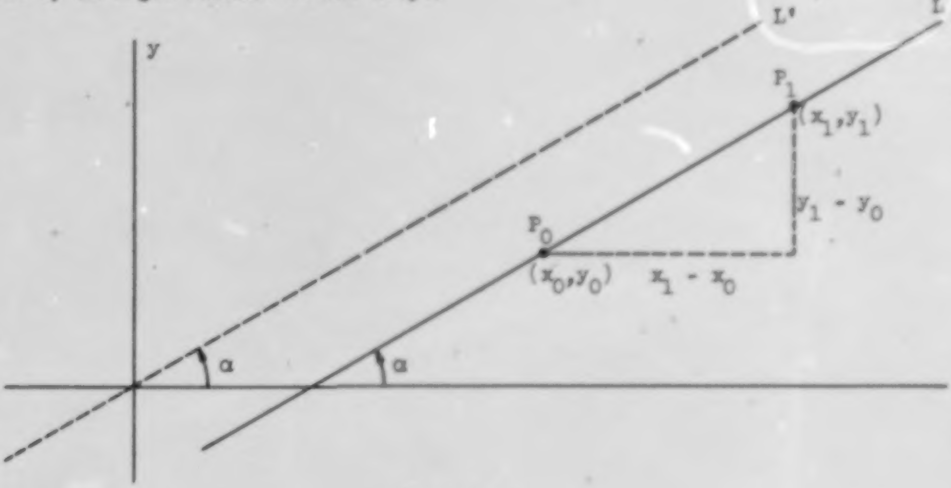


Figure 2-12

In Figure 2-12 the angle α is the angle of inclination of line L. The measure of angle α is the inclination of L. The angle α has the same measure as the corresponding angle measured in a counterclockwise direction from the positive side of the x-axis to the unique line L' which is parallel to L and contains the origin. (If L contains the origin, angle α corresponds to itself.)

We observe that if L is the x-axis or is parallel to the x-axis, its inclination is 0°. We also note that the slope of L is the tangent of angle α. If $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$, then for the line $\overleftrightarrow{P_0P_1}$

$$\tan \alpha = m = \frac{y_1 - y_0}{x_1 - x_0}.$$

For an angle α measured in degrees or radians, it is always the case that $0 \leq \alpha < 180^\circ$ or $0 \leq \alpha < \pi$, respectively.

Example 5.

(a) If the slope of a line is $\sqrt{3}$, then $\tan \alpha = \sqrt{3}$ and the inclination α of the line is 60° or $\frac{\pi}{3}$.

(b) For the line containing the points $(-1,4)$ and $(2,7)$

$$\tan \alpha = m = \frac{7 - 4}{2 + 1} = 1 \quad \text{and} \quad \alpha = 45^\circ \text{ or } \frac{\pi}{4}.$$

Slope-Intercept Form. The x-intercepts of any graph are the abscissas of the points of the graph which are on the x-axis. The y-intercepts are the ordinates of the points of the graph on the y-axis.

A line has a unique y-intercept if and only if its slope is defined. If the slope is defined, the line is distinct from the y-axis and is not parallel to the y-axis. The line intersects the y-axis in a single point and therefore has a unique y-intercept. If the slope is not defined, the line either is the y-axis or is parallel to the y-axis. In either case the intersection of the line and the y-axis does not contain a unique point.

Since the lines with unique y-intercepts are those for which the slope is defined, they are the same lines which have point-slope forms. The point-slope form

$$y - y_0 = m(x - x_0)$$

is equivalent to

(5) $$y = mx + (y_0 - mx_0) .$$

We observe that the y-intercept is the y-coordinate of the point whose x-coordinate is zero. If we let $x = 0$ in Equation (5), we find that the y-intercept is $y_0 - mx_0$. It is customary to denote the y-intercept by b . With this change Equation (5) becomes

(6) $$y = mx + b ,$$

which is called the slope-intercept form of the equation.

Example 6.

(a) The line with slope 3 and y-intercept -7 is represented by the equation $y = 3x - 7$.

(b) The line represented by the equation

$$\frac{y - 2}{3} = \frac{x + 4}{7} ,$$

which is equivalent to

$$y = \frac{3}{7}x + \frac{12}{7} + 2$$

or

$$y = \frac{3}{7}x + \frac{26}{7} ,$$

has slope $\frac{3}{7}$ and y-intercept $\frac{26}{7}$.

Intercept Form. A line has a unique x-intercept if and only if it does not have zero slope. The slope is zero if and only if the line either is the x-axis or is parallel to the x-axis. The line is not the x-axis and is not parallel to the x-axis if and only if it intersects the x-axis in a single point. In this case the x-intercept is unique.

It is customary to denote a unique x-intercept by a .

If the slope of a line is defined and is not zero, both intercepts are unique. Since the x-intercept is the x-coordinate of the point whose y-coordinate is zero, we let y be zero in Equation (6) and find that the x-intercept $a = \frac{-b}{m}$. If in addition $ab \neq 0$ (neither a nor b is zero), we may transform Equation (6)

$$y = mx + b$$

to obtain

$$-\frac{mx}{b} + \frac{y}{b} = 1$$

or

$$\frac{x}{-\frac{b}{m}} + \frac{y}{b} = 1 .$$

We substitute the value of the x-intercept to obtain

$$\frac{x}{a} + \frac{y}{b} = 1 . \tag{7}$$

This is called the *intercept form* of the equation of a line.

Example 7. Find the intercept form of an equation for the line containing the points $(-1,4)$ and $(2,5)$.

Solution.

(a) The line has an equation in two-point form,

$$y - 4 = \frac{5 - 4}{2 + 1}(x + 1)$$

or

$$y - 4 = \frac{1}{3}(x + 1)$$

or

$$y = \frac{1}{3}x + \frac{13}{3} .$$

The y-intercept is $\frac{13}{3}$ and when $y = 0$, $x = -13$. Hence the x-intercept is -13 and the intercept form is

$$\frac{x}{-13} + \frac{y}{\frac{13}{3}} = 1 .$$

(b) If the intercepts are a and b , then the line contains the points $(a,0)$ and $(0,b)$. Since the slope is $\frac{5 - 4}{2 + 1} = \frac{1}{3}$, it must also be the case that

$$\frac{5 - 0}{2 - a} = \frac{1}{3} \quad \text{and} \quad \frac{5 - b}{2 - 0} = \frac{1}{3} ,$$

or

$$a = -13 \quad \text{and} \quad b = \frac{13}{3} .$$

Hence, the intercept form is

$$\frac{x}{-13} + \frac{y}{\frac{13}{3}} = 1 .$$

General Form. Each of the preceding forms of the equation of a line has certain advantages, not only because it is easy to write when certain facts about the line are known, but also because each clearly displays in its written form certain geometric properties of the line. However, none of these forms is defined for all lines.

The symmetric form

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

is not defined for a line parallel to either axis, but if we transform the equation to

$$(y_1 - y_0)(x - x_0) = (x_1 - x_0)(y - y_0) \text{ , where } x_0 \neq x_1 \text{ or } y_0 \neq y_1$$

the new equation does describe any line in the plane. In order to simplify this equation, we collect all non-zero terms in one member of the equation and identify the coefficients of x and y and the constant term.

$$(y_1 - y_0)x - (x_1 - x_0)y - x_0(y_1 - y_0) + y_0(x_1 - x_0) = 0$$

is equivalent to

$$(y_1 - y_0)x + (x_0 - x_1)y + (x_1y_0 - x_0y_1) = 0 .$$

We let

$$a = y_1 - y_0 \text{ , } b = x_0 - x_1 \text{ , and } c = x_1y_0 - x_0y_1 \text{ ,}$$

and wi[illegible]

(8) $ax + by + c = 0$, where $a^2 + b^2 \neq 0$ (that is, $a \neq 0$ or $b \neq 0$) .

Equation (8) is called a *general form* of the equation of a line. It is [illegible]so called the *general linear equation* in x and y .

Example 8. Write the equations

(a) $3x + 4y - 8 = 0$ and

(b) $ax + by + c = 0$, where $abc \neq 0$, (that is, $a \neq 0$, $b \neq 0$, and $c \neq 0$) in intercept and slope-intercept form.

Solution.

(a)
$$3x + 4y - 8 = 0$$

is equivalent to

$$\frac{3x}{8} + \frac{y}{2} = 1$$

or

$$\frac{x}{\frac{8}{3}} + \frac{y}{2} = 1 ,$$

which is in the intercept form.

The original equation is also equivalent to

$$4y = -3x + 8$$

and

$$y = -\frac{3}{4}x + 2 ,$$

which is in the slope-intercept form.

(b)
$$ax + by + c = 0 \text{ , where } abc \neq 0 ,$$

is equivalent to

$$\frac{ax}{-c} + \frac{by}{-c} = 1 \text{ , where } abc \neq 0 ,$$

and

$$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1 \text{ , where } abc \neq 0 ,$$

which is in the intercept form.

$$ax + by + c = 0 \text{ , where } abc \neq 0 \text{ ,}$$

is equivalent to

$$by = -ax - c \text{ , where } abc \neq 0 \text{ ;}$$

and

$$y = -\frac{a}{b}x - \frac{c}{b} \text{ , where } abc \neq 0 \text{ ,}$$

which is in the slope-intercept form.

From Example 8(b) we observe that when an equation of a line is expressed in general form, the x- and y-intercepts are $-\frac{c}{a}$ and $-\frac{c}{b}$ respectively if they exist and the slope of the line is $-\frac{a}{b}$ if it is defined.

The great advantage of the general form is that it can be written for any line. The only shortcoming is that the geometric properties of the line are less clearly revealed by this form.

Exercises 2-5

1. Use Equation (4) to find an equation of a line containing $(2,-3)$ and having slope 2. Put the equation in general form. If the line contains the points $(p,7)$ and $(5,q)$, find p and q.
2. Find an equation of a line with slope $-\frac{2}{3}$ and passing through $(-3,5)$. If this line contains the points $(p,7)$ and $(5,q)$, find p and q.
3. Find an equation of a line containing the point $(0,b)$ and having slope 3. If the line contains the points $(p,7)$ and $(5,q)$, find p and q.
4. Find an equation of a line containing the point $(4,5)$ and having the same slope as the line $2x - 3y = 600$. Describe the relative position of these two lines.
5. Write an equation of a line having slope k and containing the point $(a,0)$. What are the coordinates of the point where the line crosses the y-axis?

6. Write an equation representing all lines containing the origin. Are you sure every line is represented by your equation? Write the equation of the one of these lines that contains the point $(-3,5)$.

7. The coordinates of A and B are $(3,5)$ and $(-5,3)$. Segments $\overline{OA}$ and $\overline{OB}$ form a right angle at the origin. Determine the slope of each segment and try to arrive at a general conclusion that you can prove.

8. Choose $(-8,8)$ as (x_0,y_0) and write the equation $3x + 4y - 8 = 0$ in symmetric form.

9. Write an equation of the line containing the points $(-4,8)$ and $(2,3)$. Exhibit the result in all seven forms so far discussed. What is the slope? what are the intercepts?

10. Write the equation $ax + by + c = 0$ in the slope-intercept form. What is the geometric interpretation of $ax + by + c = 0$,
    (a) when $b = 0$, $ac \neq 0$?
    (b) when $a = 0$, $bc \neq 0$?
    (c) when $c = 0$, $ab \neq 0$?

11. Find an equation of a line satisfying the following conditions:
    (a) Containing the point $(3,-2)$ and having y-intercept 5 .
    (b) Containing the point $(3,-2)$ and having x-intercept 5 .
    (c) Containing the midpoint of $\overline{AB}$ where $A = (-7,2)$, $B = (3,4)$. and with the same slope as the line OA .
    (d) Containing the point $(2,-4)$ and with inclination 135° .
    (e) Containing the point $(-1,-3)$ and with inclination 30° .

12. In triangle ABC ; $A = (1,-2)$, $B = (3,2)$, $C = (0,4)$. Find an equation of each of the following lines:
    (a) $\overleftrightarrow{AB}$.
    (b) The median from A .
    (c) The line joining the midpoints of $\overline{AC}$ and $\overline{BC}$.

13. Find an equation of a line containing the point $P = (5,8)$ which forms with the coordinate axes a triangle with area 10 square units.

Review Exercises--Section 2-1 through Section 2-5

In Exercises 1-4 find the graph of the sets described on a line with a linear coordinate system.

1. $\{x : 1 < x \leq 2\}$.

2. $\{x : (x - 1)(x + 2) = 0\}$.

3. $\{x : |x| < 3\}$.

4. $\{x : |x - 4| \geq 2\}$.

In Exercises 5 to 9 graph and describe the geometric representation in one-space and 2-space.

5. $\{x : x + 4 = 0\}$.

6. $\{x : |x| + 4 = 0\}$.

7. $\{x : 2 < x < 6\}$.

8. $\{x : 2 \leq |x|\}$.

9. $\{x : |x| \leq 6\}$.

10. Find the midpoints and trisection points of

    (a) $\overline{AB} = \{x : -1 \leq x \leq 2\}$.

    (b) $\overline{BC} = \{x : |x + 2| \leq 3\}$.

    (c) $\overline{CD} = \{x : c \leq x \leq d \text{ , } (c + 2)(d - 3) = 0\}$.

11. Find a polar representation for the points whose rectangular coordinates are:

    (a) $(1,\sqrt{3})$.

    (b) $(-\sqrt{2},\sqrt{2})$.

    (c) $(3,-4)$.

    (d) $(-2,-3)$.

    (e) $(1,0)$.

    (f) $(0,1)$.

12. Find the rectangular representation for the points whose polar coordinates are:

    (a) $(4,60^\circ)$.

    (b) $(3,\frac{2\pi}{3})$.

    (c) $(-2,\frac{5\pi}{4})$.

    (d) $(6,\frac{9\pi}{4})$.

    (e) $(5,-135^\circ)$.

    (f) $(-3,-750^\circ)$.

In each exercise from 13 to 18 write an equation of a line which satisfies the given conditions.

13. Contains $(-2,5)$; $m = -\frac{3}{4}$.

14. Contains $(-3,2)$, $(8,10)$.

15. Contains $(-4,-5)$, $(-6,-10)$.

16. Contains $(4,5)$; $\alpha = 120^\circ$.

17. Horizontal; y-intercept 6 .

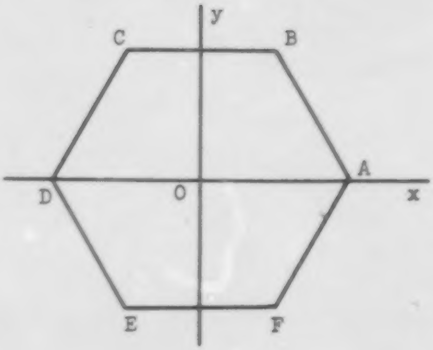
18. Vertical; x-intercept 4 .

Exercises 19 - 25 refer to the figure at the right, which represents a regular hexagon with sides of length 6 . The coordinates of the vertices are:

$A = (6,0)$; $B = (3,3\sqrt{3})$;

$C = (-3,3\sqrt{3})$; $D = (-6,0)$;

$E = (-3,-3\sqrt{3})$; $F = (3,-3\sqrt{3})$.



19. Write equations of the lines determined by each of the six sides in slope-intercept form.

20. Write equations of the lines determined by each of the six sides in general form.

21. Write equations of the lines determined by each of the six sides in symmetric form.

22. Find the slopes of $\overleftrightarrow{AC}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{DF}$.

23. Find the coordinates of the two trisection points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$.

24. Find coordinates of the points P , Q , and R , where

(a) P is on $\overleftrightarrow{AB}$ and $\frac{d(A,P)}{d(A,B)} = \frac{2}{3}$ (two answers).

(b) Q is on $\overleftrightarrow{BC}$ and $\frac{d(B,Q)}{d(Q,C)} = \frac{3}{4}$ (two answers).

(c) R is on $\overleftrightarrow{CD}$ and $\frac{d(C,R)}{d(R,D)} = \frac{4}{5}$ (two answers).

25. Find the inclination, to the nearest degree, of $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{AF}$.

26. Summarize the different forms of the equation of a line in a table, listing for each form its particular advantages and disadvantages.

Which form, or forms, of equations for a line would you use to answer each of the following questions in the most efficient way? Be prepared to explain your answer.

(a) Is the point $(3,7)$ on the line?

(b) Does the line intersect the x-axis? If so, where?

(c) Does the line contain the origin?

(d) What is the slope of the line?

(e) Find the ordinate of the point where the abscissa is 5 .

(f) Find the point on the line where the two coordinates are equal.

(g) If the point $(3,3-k)$ is on the line, find k .

(h) Suppose the point P is on the line; find the points R and S on the line which are 5 units from P .

Graph the relations of Exercises 27 to 32.

27. $\{(x,y) : |x| + |y| - 10 = 0\}$.

28. $\{(x,y) : |x| + |y| = 0\}$.

29. $\{(x,y) : x - y < 1\}$.

30. $\{(x,y) : x - y \leq 1\}$.

31. $\{(x,y) : x - y < 1\} \cap \{(x,y) : x + y < 1\}$.

32. $R_1 = \{(x,y) : |x| < 4\}$, $R_2 = \{(x,y) : |y| < 4\}$, $R_3 = R_1 \cap R_2$.

33. Discuss Exercise 32 if $<$ is changed to $\leq$. What geometric interpretation can you give for $R_1 \cup R_2$?

34. Two thermometers in common use are the Fahrenheit and Centigrade. The freezing point for water is $32^{\circ}F$ and $0^{\circ}C$; the boiling point for water is $212^{\circ}F$ and $100^{\circ}C$. Derive a formula for expressing temperature on one scale in terms of the other. Find the temperature reading which gives the same number on both scales.

35. Graph the following relations:

(a) $R_1 = \{(x,y) : 2x + 3y - 6 = 0\}$.

(b) $R_2 = \{(x,y) : 7x + y - 2 = 0\}$.

(c) $R_3 = \{(x,y) : 5x - 2y - 15 = 0\}$.

(d) $R_4 = \{(x,y) : 2x + 3y \leq 6\}$.

(e) $R_5 = \{(x,y) : 7x + y \geq 2\}$.

(f) $R_6 = \{(x,y) : 5x - 2y \leq 15\}$.

(g) $R_4 \cap R_5 \cap R_6$.

Challenge Exercises

Note: The symbol $[x]$ is used to represent the first integer $\leq x$, or stated in another way, $[x]$ means the greatest integer not greater than x . For instance, if $0 < x < 1$, $[x] = 0$; if $x = 2$, $[x] = 2$; if $-1 < x < 0$, $[x] = -1$.

Graph the relations.

1. (a) $R_1 = \{(x,y) : [x] = x\}$.

   (b) $R_2 = \{(x,y) : [y] = y\}$.

   (c) $R_3 = \{(x,y) : [x] = x\} \cap \{(x,y) : [y] = y\}$.

   (d) $R_4 = \{(x,y) : [x] = x\} \cup \{(x,y) : [y] = y\}$.

   (e) $R_5 = \{(x,y) : [x] = [y]\}$.

   (f) $R_6 = \{(x,y) : [x] = [y + k]\}$.

   (g) $R_7 = \{(x,y) : [x] = [-y]\}$.

   (h) $R_8 = \{(x,y) : [x] = -[y]\}$.

2. Graph $r = \theta$.

3. Graph $r^2 = \theta$.

4. When we introduced a system of rectangular coordinates into a plane, we used on each axis linear coordinate systems in the same units. Then if $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are any two points in the plane,

$$d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Suppose instead that on the x- and y-axes we use linear coordinate systems for which the units are in the ratio r to s respectively, where $r \neq s$.

(a) Find a formula for $d(P_1,P_2)$ in the units of the x-axis.

(b) Find a formula for $d(P_1,P_2)$ in the units of the y-axis.

(c) Let P, Q, R, and S be four points in the plane, with coordinates (p_1,p_2), (q_1,q_2), (r_1,r_2), and (s_1,s_2) respectively. Under what conditions is $\overline{PQ} \cong \overline{RS}$ and

$$\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} = \sqrt{(r_1 - s_1)^2 + (r_2 - s_2)^2} \text{ ?}$$

5. Find the graph of $S = \{(x,y) : (4x + 3y - 5)^2 = 0\}$. Can you find a simpler analytic representation for the graph?

6. What is the graph of $T = \{(x,y) : (ax + by + c)^k = 0$, where $a^2 + b^2 \neq 0$ and k is a positive integer$\}$? Can you find a simpler analytic representation for the graph?

7. Find the intersection of $L_1 = \{(x,y) : 3x + 2y - 1 = 0\}$ and $L_2 = \{(x,y) : 2x - 3y + 2 = 0\}$.

8. Find the graph of $U = \{(x,y) : (3x + 2y - 1)(2x - 3y + 2) = 0\}$.

9. Find the graph of $V = \{(x,y) : (x + y)(x - y) = 0\}$.

10. Find the graph of $W = \{(x,y) : xy = 0\}$.

11. Assume that $L_0 = \{(x,y) : a_0x + b_0y + c_0 = 0 , a_0^2 + b_0^2 \neq 0\}$ and $L_1 = \{(x,y) : a_1x + b_1y + c_1 = 0 , a_1^2 + b_1^2 \neq 0\}$ have a unique point (x_1,y_1) in common. What can you say about x_1 and y_1 if a_0, a_1, b_0, b_1, c_0, and c_1 are

    (a) integral?
    (b) rational?
    (c) real?
    (d) complex?

12. What can you say about the graph of

    (a) $R = \{(x,y) : (3x - 2y + 2) + k(x + y + 1) = 0$, where k is constant$\}$?
    (b) $S = \{(x,y) : (x + y + 1) + k(3x - 2y + 2) = 0$, where k is constant$\}$?
    (c) $T = \{(x,y) : m(3x - 2y + 2) + n(x + y + 1) = 0$, where $m^2 + n^2 \neq 0$, and m and n are contant$\}$?

13. What can you say about the graph of

(a) $U = \{(x,y) : (3x - 2y + 2) + t(x + y + 1) = 0$, where t is a real variable$\}$?

(b) $V = \{(x,y) : (x + y + 1) + t(3x - 2y + 2) = 0$, where t is a real variable$\}$?

(c) $W = \{(x,y) : s(3x - 2y + 2) + t(x + y + 1) = 0$, where $s^2 + t^2 \neq 0$, and s and t are real variables$\}$?

14. Assume that the linear equations $a_0x + b_0y + c_0 = 0$, where $a_0^2 + b_0^2 \neq 0$, and $a_1x + b_1y + c_1 = 0$, where $a_1^2 + b_1^2 \neq 0$, are not equivalent. What can you say about the graph of

(a) $R = \{(x,y) : (a_0x + b_0y + c_0) + k(a_1x + b_1y + c_1) = 0$, where k is constant$\}$?

(b) $S = \{(x,y) : (a_1x + b_1y + c_1) + k(a_0x + b_0y + c_0) = 0$, where k is constant$\}$?

(c) $T = \{(x,y) : (a_0x + b_0y + c_0) + t(a_1x + b_1y + c_1) = 0$, where t is real$\}$?

(d) $U = \{(x,y) : (a_1x + b_1y + c_1) + t(a_0x + b_0y + c_0) = 0$, where t is real$\}$?

(e) $V = \{(x,y) : m(a_0x + b_0y + c_0) + n(a_1x + b_1y + c_1) = 0$, where $m^2 + n^2 \neq 0$, and m and n are constant$\}$?

(f) $W = \{(x,y) : s(a_0x + b_0y + c_0) + t(a_1x + b_1y + c_1) = 0$, where $s^2 + t^2 \neq 0$, and s and t are real variables$\}$?

15. What is the graph of

(a) $S = \{(x,y) : 0 = 1\}$?

(b) $T = \{(x,y) : 1 = 1\}$?

2-6. Direction on a Line.

Although there are two senses of direction implicit in our intuitive notion of a line, neither one is dominant or primary. When we represent a line analytically, we may suggest a specific sense of direction for the line. When we undertake a geometric description of the line in terms of an associated angle, we suggest a sense of direction for the line if a side of the angle is contained in the line.

In this section we shall introduce some of the analytic ideas and terms which may be used once a sense of direction has been assumed for a line. We shall also consider the geometric interpretation of the ideas.

When we speak of the line segment from P_0 to P_1, we suggest a sense of direction on the line. If $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$, the numbers $\ell = x_1 - x_0$ and $m = y_1 - y_0$ also suggest this sense of direction.

The numbers ℓ and m are called direction numbers of L. For the ordered pair of direction numbers we use the symbol (ℓ,m). Since this symbol is also used for a point, care must be exercised to avoid ambiguity. Clearly a line has infinitely many pairs of direction numbers, since there are infinitely many pairs of points P_0 and P_1 which determine it. However, all the pairs for a given line L are related in a very simple way. If L has a slope and (ℓ,m) and (ℓ',m') are two pairs of direction numbers for L, then $\frac{m}{\ell} = \frac{m'}{\ell'}$ and there is a number $c \neq 0$ such that $\ell' = c\ell$ and $m' = cm$. If L has no slope, there is still such a number c, though the argument above does not prove it. If two lines are parallel, a similar argument shows that any two pairs of direction numbers for the two are related in the same way. Thus it is natural to make the following definition:

> DEFINITION. The pair (ℓ,m) of direction numbers is said to be equivalent to the pair (ℓ',m') if and only if there is a number $c \neq 0$ such that $\ell' = c\ell$, $m' = cm$.

The preceding discussion can now be summarized in the following statement.

> Two distinct lines in a plane are parallel if and only if any pair of direction numbers for one is equivalent to any pair for the other.

A pair (ℓ,m) of direction numbers for a line L may be said to determine a direction on the line in the following sense. Let $P_0 = (x_0,y_0)$ be a fixed point of L and $P = (x,y)$ any other point of L. Then $x - x_0 = c\ell$ and $y - y_0 = cm$, or

$$x = x_0 + c\ell ,$$

$$y = y_0 + cm , \quad \text{where } c \neq 0 .$$

The point P_0 separates L into two sets of points; the points on one side of P_0 are given by positive values of c. P_0 and the points of L given by positive values of c form a ray, which we call the positive ray (on L) with endpoint P_0. If $P_1 = (x_1,y_1)$ is another point of L, then P_1 and the points $P = (x,y)$ given by

$$x = x_1 + c\ell ,$$

$$y = y_1 + cm , \quad \text{where } c > 0 ,$$

form another positive ray on L. The intersection (set of common points) of the positive rays with endpoints P_0 and P_1 is one of those two rays. Intuitively speaking, all the positive rays point in the same direction on L. The pair $(c\ell,cm)$ of direction numbers determines the same direction on L as (ℓ,m) if and only if $c > 0$.

If (ℓ,m) is a pair of direction numbers for L, the equivalent pair

$$(\lambda,\mu) = \left(\frac{\ell}{\sqrt{\ell^2 + m^2}}, \frac{m}{\sqrt{\ell^2 + m^2}}\right)$$

is of particular importance. Such a pair is sometimes called a normalized pair. You should observe that $\lambda^2 + \mu^2 = 1$.

Let L be a line in a plane with a rectangular coordinate system and let L' be the line parallel to L which passes through the origin. (If L contains the origin, L' = L.) Then L and L' have the same pair of direction numbers (ℓ,m). Figure 2-13a shows the situation if $\ell > 0$ and $m > 0$, Figure 2-13b if $\ell > 0$ and $m < 0$, Figure 2-13c if $\ell < 0$ and $m < 0$, and Figure 2-13d if $\ell < 0$ and $m > 0$.

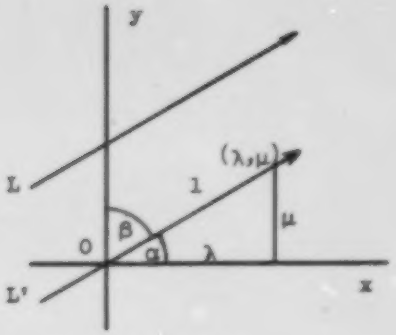


Figure 2-13a

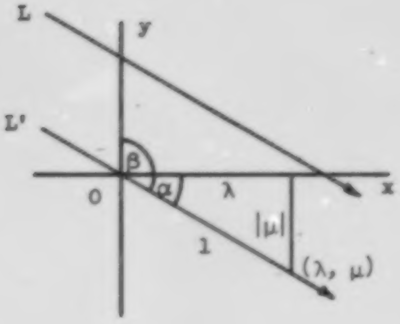


Figure 2-13b

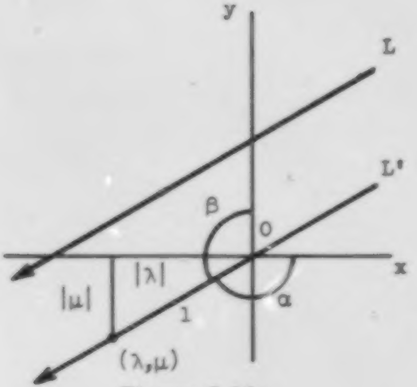


Figure 2-13c

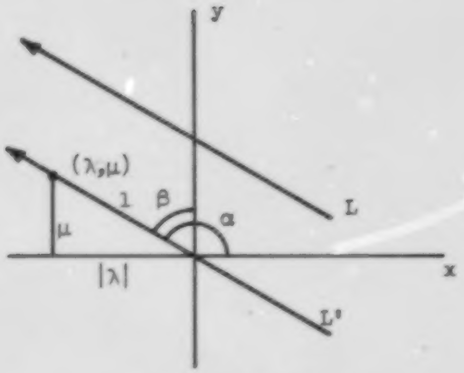


Figure 2-13d

The arrowheads show the positive directions on L and L'. The angles α and β are called the direction angles of the line L with the positive direction determined by the pair (ℓ,m) of direction numbers. $\angle\alpha$ is the angle formed by the positive ray on L' with the origin as endpoint, and the positive half of the x-axis. $\angle\beta$ is the angle formed by the positive ray on L' with the origin as endpoint, and the positive half of the y-axis. We note that the direction angles are geometric angles, with the single exception that their sides may be collinear. Hence, $0 \le \alpha \le 180^\circ$ and $0 \le \beta \le 180^\circ$.

If $c > 0$, each equivalent pair $(c\ell,cm)$ of direction numbers for L is also the pair of coordinates for a point on L'. The point with (λ,μ), the normalized pair, as coordinates has been indicated in each case of Figure 2-13. Consideration of these cases reveals that, since $\lambda^2 + \mu^2 = 1$,

$\cos\alpha = \lambda$, and $\cos\beta = \mu$. The cosines of direction angles of a line L are called direction cosines for the line.

The direction numbers, angles, and cosines of a ray R are defined to be the direction numbers, angles, and cosines, respectively, of the line containing R with positive direction determined by R .

Example 1. What are the pairs of direction numbers for the line determined by the points $P_0 = (-2,7)$ and $P_1 = (6,-2)$?

Solution. One pair is $\big(-2 - 6, 7 - (-2)\big)$, or $(-8,9)$, but any equivalent pair $(-8c,9c)$, where $c \neq 0$, will do. Since any pair (ℓ,m) must be such that $\frac{\ell}{m} = \frac{-8}{9}$ or $9\ell + 8m = 0$, we may write this as

$\{(\ell,m) : 9\ell + 8m = 0 , \ell^2 + m^2 \neq 0\}$.

Example 2.

(a) What are the direction cosines and the measures of the direction angles for the line L with the positive direction determined by the pair $(1,1)$ of direction numbers?

(b) What are the direction cosines and angles for the same line L , but with the positive direction determined by the equivalent pair $(-1,-1)$?

Solution.

(a) $\cos\alpha = \lambda = \dfrac{\ell}{\sqrt{\ell^2 + m^2}}$ and $\cos\beta = \mu = \dfrac{m}{\sqrt{\ell^2 + m^2}}$.

Therefore, $\cos\alpha = \frac{1}{\sqrt{2}}$, $\cos\beta = \frac{1}{\sqrt{2}}$ and $\alpha = \beta = 45^\circ$.

(b) In this case, $\cos\alpha = \frac{-1}{\sqrt{2}}$, $\cos\beta = \frac{-1}{\sqrt{2}}$ and $\alpha = \beta = 135^\circ$.

Example 3. Find the direction angles and direction cosines of the line through $(1,2)$ with positive direction determined by the pair $(-\sqrt{3},1)$ of direction numbers. Do the same when the positive direction is determined by the pair $(\sqrt{3},-1)$.

Solution. In the first case, $\lambda = -\frac{\sqrt{3}}{2}$ and $\mu = \frac{1}{2}$. Since by definition $0 \leq \alpha \leq 180^\circ$ and $0 \leq \beta \leq 180^\circ$, and since $\cos \alpha = \lambda$ and $\cos \beta = \mu$, we see that $\alpha = 150^\circ$, $\beta = 60^\circ$. If we consider the other direction on L, we have $\cos \alpha = \frac{\sqrt{3}}{2}$, $\cos \beta = -\frac{1}{2}$. Hence $\alpha = 30^\circ$, $\beta = 120^\circ$.

Examples 2 and 3 suggest a careful distinction to be made. A line has unsensed direction, or perhaps it would be better to say that two opposite senses of direction are implied for a given line, but neither one is dominant. Some of the pairs of direction numbers for a line imply each sense, but if we select a single pair, we select a single sense of direction as well. Direction angles and direction cosines are defined only for a line with a specified sense of direction. We shall call such a line a directed line. The sense of direction may be specified by the context, such as the choice of a single pair of direction numbers for the line.

In Figure 2-14 we observe that either $\angle\alpha$ and $\angle\beta$ or $\angle\alpha'$ and $\angle\beta'$ might be direction angles for line L. Since $\alpha + \alpha' = 180^\circ$ and $\beta + \beta' = 180^\circ$, we note that $\cos \alpha' = -\cos \alpha$ and $\cos \beta = -\cos \beta$. Thus, if the normalized pair (λ, μ) of direction numbers are direction cosines for a directed line, $(-\lambda, -\mu)$ are the pair of direction cosines for the same line with opposite direction; if $\angle\alpha$ and $\angle\beta$ are direction angles for a directed line, their supplements are direction angles for the same line with opposite direction.

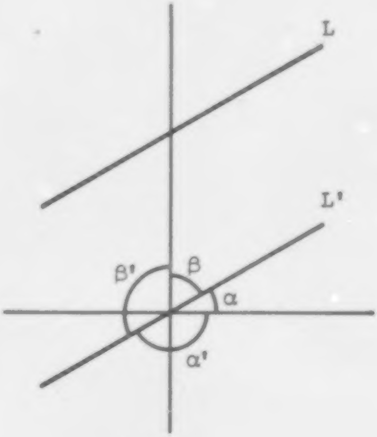


Figure 2-14

Example 4. Find direction numbers, cosines, and angles for the lines

(a) $\{(x,y) : 3x - 4y - 5 = 0\}$ and

(b) $\{(x,y) : ax + by + c = 0\}$, $b \neq 0\}$.

Solution.

(a) We observe that if a nonvertical line has a pair (ℓ,m) of direction numbers and an equation in general form, $ax + by + c = 0$, then the slope of the line is given by both $\frac{m}{\ell}$ and $-\frac{a}{b}$. Therefore

$$\frac{m}{\ell} = -\frac{a}{b}, \text{ where } \ell b \neq 0 .$$

Since $3x - 4y - 5 = 0$ is in general form, the slope of the line is $\frac{3}{4}$, $(4,3)$ is a pair of direction numbers, and any other pair $(4c,3c)$, where $c \neq 0$, is an equivalent pair of direction numbers. The normalized pair (λ,μ) of direction numbers, or direction cosines $\cos \alpha$ and $\cos \beta$, is either

$$\left(\frac{4}{\sqrt{4^2 + 3^2}}, \frac{3}{\sqrt{4^2 + 3^2}}\right) = \left(\frac{4}{5}, \frac{3}{5}\right) \text{ or } \left(-\frac{4}{5}, -\frac{3}{5}\right),$$

depending on which sense of direction is adopted for the line. We use tables of trigonometric functions to discover that the measures α and β of the corresponding direction angles are (approximately) 37° and 53°, or 143° and 127° respectively.

(b) For the general form of an equation of a line $ax + by + c = 0$, where $b \neq 0$, the slope is $-\frac{a}{b}$. Thus, $(-b,a)$, $(b,-a)$, and, in general, $(-bk,ak)$, where $k \neq 0$, are pairs of direction numbers. The normalized pair, or pair of direction cosines, is

$$\left(\frac{-b}{\sqrt{a^2 + b^2}}, \frac{a}{\sqrt{a^2 + b^2}}\right) \text{ or } \left(\frac{b}{\sqrt{a^2 + b^2}}, \frac{-a}{\sqrt{a^2 + b^2}}\right),$$

depending on the sense of direction. Once the direction cosines are found, the direction angles are uniquely determined, since by definition $0 \leq \alpha \leq 180^\circ$ and $0 \leq \beta \leq 180^\circ$.

Example 5. Consider the line $L = \{(x,y) : \frac{x}{a} + \frac{y}{b} = 1 \text{ , } ab \neq 0\}$. Let O be the origin; let A and B be the points of L on the x- and y-axes respectively.

(a) Write an equation of L in general form.

(b) Find the length of the altitude $\overline{OC}$ on the hypotenuse of right triangle AOB .

(c) Find the direction cosines of $\overrightarrow{OC}$.

(d) How are the coefficients in the answer to Part (a) related to the results of Parts (c) and (b)?

Solution.

(a) $\frac{x}{a} + \frac{y}{b} = 1$ is equivalent to $bx + ay - ab = 0$, which is in general form.

(b) The area of $\triangle AOB$ is equal both to $\frac{1}{2}|ab|$ and to $\frac{1}{2}\sqrt{a^2 + b^2} \cdot d(O,C)$; hence, $\frac{1}{2}|ab| = \frac{1}{2}\sqrt{a^2 + b^2} \cdot d(O,C)$.

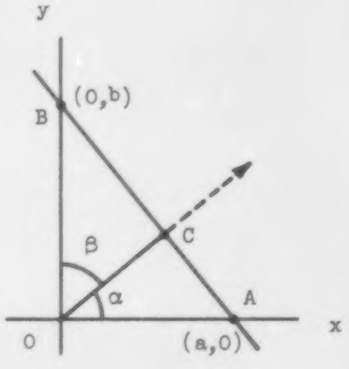
Therefore, the length of $\overline{OC} = d(O,C) = \dfrac{|ab|}{\sqrt{a^2 + b^2}}$.

(c) $\cos \alpha = \cos \angle ABO = \dfrac{b}{\sqrt{a^2 + b^2}}$. (Why?)

$\cos \beta = \cos \angle BAO = \dfrac{a}{\sqrt{a^2 + b^2}}$. (Why?)

(d) Lastly, we note that the results of Parts (c) , and (b) apart from a possible difference in sign, are proportional to the coefficients in the equation obtained in Part (a). The constant of proportionality is

$$\frac{1}{\sqrt{a^2 + b^2}} \left(\text{or } \frac{-1}{\sqrt{a^2 + b^2}}\right).$$



Exercises 2-6

1. Find pairs of direction numbers for the line through each pair of points given below. Use both possible orders.

(a)	(5,-1) , (2,3)	(e)	(1,1) , (2,2)
(b)	(0,0) , (4,1)	(f)	(-1,-1) , (1,1)
(c)	(2,-3) , (2,3)	(g)	(1,0) , (0,1)
(d)	(-1,4) , (-6,4)	(h)	(2,-2) , (-2,2)

2. Find the normalized pairs of direction numbers for the lines in Exercise 1.

3. Find the direction angles of the lines in Exercises 1 and 2.

4. Given the pairs (3,-4) , (2,0) , (0,-3) , (-1,2) , and (-2,1) of direction number,
   (a) find the slope of a line with each pair as a pair of direction numbers
   (b) find a pair equivalent to each pair, and find the corresponding direction angles
   (c) draw the line through the origin with each pair as its direction numbers, and indicate the positive direction on each line determined by the pair (Do not draw too many on one sketch.)
   (d) indicate on your sketches the direction angles of each directed line.

5. Let $P_0 = (x_0,y_0)$, $P_1 = (x_0,y_1)$, and $P_2 = (x_0,y_2)$ be any three distinct points on a line parallel to the y-axis in a plane with a rectangular coordinate system. Show that the pair of direction numbers determined by P_0 and P_1 and the pair of direction numbers determined by P_0 and P_2 are equivalent.

6. Let α and β be the direction angles of the line L with positive direction determined by the pair (ℓ,m) of direction numbers, α' and β' the direction angles of L with positive direction determined by the pair $(-\ell,-m)$ of direction numbers. Prove that α and α' are supplementary, and that β and β' are supplementary.

7. Assume that in each part of Figure 2-13 a polar coordinate system has also been introduced in the usual way. Let ω denote the measure of a polar angle which contains the positive ray of L' with endpoint at the origin.
   (a) Show that in each case $\sin \omega = \cos \beta$.
   (b) Show that $\sin \omega = \cos \beta$ for any positive ray lying on an axis.

8. Find pairs of direction numbers, direction cosines, and direction angles for the lines L , M , and N , where

(a) $L = \{(x,y) : x - 2y + 7 = 0\}$

(b) $M = \{(x,y) : y = -\frac{1}{2}x + 7\}$

(c) $N = \{(x,y) : \frac{x}{6} - \frac{y}{5} = 1\}$.

2-7. The Angle Between Two Lines; Parallel and Perpendicular Lines.

We have developed verious forms of an equation of a line. Here we shall use equations to answer a question about the lines they represent: What angle is formed by two lines? In particular, are two lines perpendicular or parallel?

We observed that the slope of lines parallel to the x-axis is zero, and that lines parallel to the y-axis have no slope. Because of the customary orientation of the axes we usually refer to lines parallel to the x-axis as horizontal lines and to lines parallel to the y-axis as vertical lines.

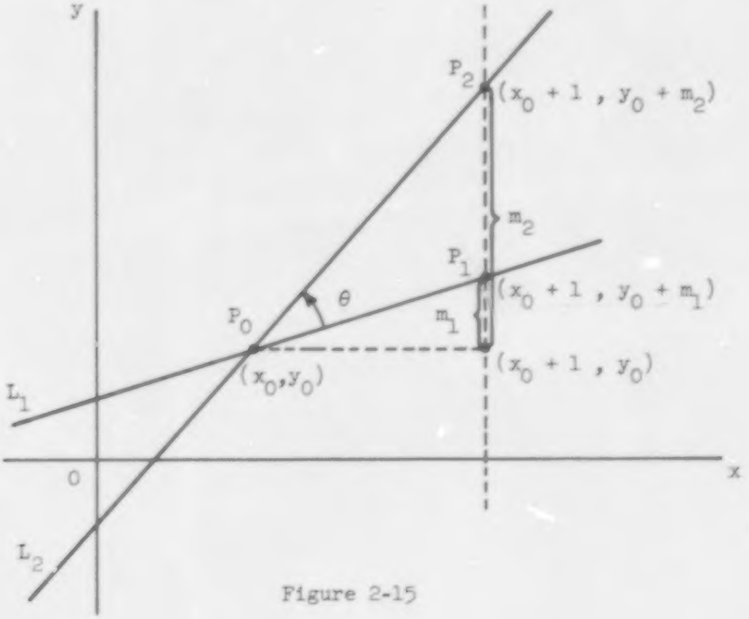


Figure 2-15

In Figure 2-15 we indicate two nonvertical lines L_1 and L_2, intersecting at the point $P_0 = (x_0,y_0)$. The vertical line represented by the equation $x = x_0 + 1$ will intersect these lines at P_1 and P_2 respectively. If we represent the slopes of L_1 and L_2 by m_1 and m_2 respectively, the coordinates of P_1 and P_2 will be $(x_0 + 1, y_0 + m_1)$ and $(x_0 + 1, y_0 + m_2)$ respectively. If in triangle $P_0P_1P_2$ we apply the distance formula and the Law of Cosines in terms of $\angle P_1P_0P_2 = \angle\theta$, we obtain

$$\Big(d(P_1,P_2)\Big)^2 = \Big(d(P_0,P_1)\Big)^2 + \Big(d(P_0,P_2)\Big)^2 - 2d(P_0,P_1)d(P_0,P_2)\cos\theta ,$$

or

$$(m_2 - m_1)^2 = 1 + m_1^2 + 1 + m_2^2 - 2\sqrt{1 + m_1^2}\cdot\sqrt{1 + m_2^2}\cos\theta .$$

This is equivalent to

$$-2m_1m_2 = 2 - 2\sqrt{1 + m_1^2}\cdot\sqrt{1 + m_2^2}\cos\theta ,$$

or

(1) $$\cos\theta = \frac{1 + m_1m_2}{\sqrt{1 + m_1^2}\sqrt{1 + m_2^2}} .$$

Example 1. Find the measures of the angles of intersection between the lines represented by the equations $y = \frac{1}{3}x + 1$ and $y = 2x + 1$.

Solution. Since the equations are in slope-intercept form, we perceive immediately that the slopes of the lines are $\frac{1}{3}$ and 2. We substitute these values in Equation (1) to obtain

$$\cos\theta = \frac{1 + (\frac{1}{3})(2)}{\sqrt{1 + (\frac{1}{3})^2}\cdot\sqrt{1 + 2^2}} = \frac{\frac{5}{3}}{\frac{\sqrt{10}}{9}\cdot\sqrt{5}} = \frac{\frac{5}{3}}{\frac{5}{3}\sqrt{2}} = \frac{1}{\sqrt{2}} .$$

Thus $\theta = 45^\circ$, and the other three angles of intersection will have measures of 45°, 135°, and 135°

In your previous courses you discovered that two nonvertical lines are parallel or the same if and only if they have the same slope. Clearly all vertical lines are parallel. You also discovered that two nonvertical lines are perpendicular if and only if the product of their slopes is -1 . It should be clear that a vertical line is perpendicular to a second line if and only if the second line is horizontal.

In Equation (1) we note that the lines are perpendicular if and only if

$$\cos\theta = 0, \text{ or } m_1m_2 = -1 . \qquad (2)$$

Example 2. Find an equation for the line L which contains the point $P = (4,3)$ and which is perpendicular to the line represented by the equation $2x + 3y + 7 = 0$.

Solution 1. In the previous section we observed that the slope of a line represented by an equation with general form $ax + by + c = 0$, $(b \neq 0)$, is $-\frac{a}{b}$. Thus the line above has slope $-\frac{2}{3}$. If L is perpendicular to the given line, its slope m must be such that

$$-\frac{2}{3}m = -1 \text{ , or } m = \frac{3}{2} .$$

Since L contains $P = (4,3)$, it has the equation in point-slope form,

$$(y - 3) = \frac{3}{2}(x - 4) .$$

This is equivalent to

$$\frac{3}{2}x - y - 3 = 0 ,$$

or

$$3x - 2y - 6 = 0 .$$

Solution 2. We might have developed a more general equation for a line L which contains $P_0 = (x_0,y_0)$, and which is perpendicular to a line with equation $ax + by + c = 0$, $(ab \neq 0)$. We observe that the slope m of L must be such that

$$-\frac{a}{b}m = -1 \text{ , or } m = \frac{b}{a} .$$

Thus L must have the equation in point-slope form,

$$y - y_0 = \frac{b}{a}(x - x_0) .$$

This is equivalent to

$$(3) \qquad bx - ay - (bx_0 - ay_0) = 0 .$$

If we substitute the specific values for a , b , x_0 , and y_0 in this general equation, we obtain

$$3x - 2y - (3 \cdot 4 - 2 \cdot 3) = 0 \text{ , or } 3x - 2y - 6 = 0 .$$

If we generalize the notion of angle so that we may speak meaningfully of the measure of the "angle" between two parallel lines, we may obtain both these results as corollaries to the more general problem of determining the angle between two lines. Let two parallel directed lines have the same sense of direction. Then the projection of each positive ray of one line on the second line is also a ray and coincides with a positive ray of the second line. The coincident rays form angles whose measure is 0° or 0 radians. When two parallel directed lines have opposite senses of direction, the projection of each positive ray of one line on the second line is also a ray, but in this case, it is opposite to a positive ray of the second line. The pairs of opposite rays form angles whose measure is 180° or π radians. We speak of parallel and antiparallel directed lines respectively to distinguish between these two cases.

The preceding discussion suggests the following conventions. The measure of the angle between two parallel directed lines is said to be 0° or 0 radians. The measure of the angle between two antiparallel lines is said to be 180° or π radians.

Although the Law of Cosines was not developed for angles of measure 0° or 180° , the relationship it describes is still valid. We shall leave the justification as an exercise. If this extension is made, we may apply Equation (1) to parallel and antiparallel directed lines. In these cases, equivalent conditions are that $\cos\theta = 1$ and $\cos\theta = -1$ respectively. Thus, if the lines are parallel, $\cos\theta = \pm 1$ and Equation (1) becomes

$$\frac{1 + m_1m_2}{\sqrt{1 + m_1^2}\sqrt{1 + m_2^2}} = \pm 1 .$$

This is equivalent to

$$(1 + m_1m_2)^2 = (1 + m_1^2)(1 + m_2^2) ,$$

or

$$1 + 2m_1m_2 + m_1^2m_2^2 = 1 + m_1^2 + m_2^2 + m_1^2m_2^2 .$$

This becomes

$$m_1^2 - 2m_1m_2 + m_2^2 = 0 ,$$

or

$$(m_1 - m_2)^2 = 0 ,$$

which is true if and only if $m_1 = m_2$. Thus, nonvertical lines are parallel if and only if

(4) $\cos \theta = \pm 1$, which is equivalent to $m_1 = m_2$.

Thus, we may express the condition that two nonvertical lines are parallel either in terms of the angle between them or in terms of their slopes.

Example 3. Write an equation in general form for

(a) the line containing the point $(1,2)$ and parallel to the line $L = \{(x,y) : 3x - 2y + 6 = 0\}$, and

(b) the line containing (x_0,y_0) and parallel to the line $L = \{(x,y) : ax + by + c = 0$, where $b \neq 0\}$.

Solutions.

(a) The slope of both lines must be $\frac{3}{2}$, so the required line must have as an equation in point-slope form,

$$y - 2 = \frac{3}{2}(x - 1) .$$

This is equivalent to

$$2y - 4 = 3x - 3 , \text{ or } 3x - 2y + 1 = 0 .$$

(b) The slope of both lines must be $-\frac{a}{b}$, so the required line must have as an equation in point-slope form,

$$y - y_0 = -\frac{a}{b}(x - x_0) .$$

This is equivalent to

$$by - by_0 = -ax + ax_0 ,$$

or

(5) $$ax + by - (ax_0 + by_0) = 0 .$$

Since equations representing lines are frequently given in general form, we write an equivalent expression to Equation (1) for the cosine of the angle between two lines in terms of the coefficients in the equations.

Let two nonvertical lines L_1 and L_2 have respective slopes m_1 and m_2 and be represented by the equations

$$a_1x + b_1y + c_1 = 0 , \text{ where } a_1^2 + b_1^2 \neq 0 ,$$

and

$$a_2x + b_2y + c_2 = 0 , \text{ where } a_2^2 + b_2^2 \neq 0 .$$

We have observed that

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2} .$$

If we substitute these values in Equation (1), we obtain

$$\cos\theta = \frac{1 + \frac{a_1a_2}{b_1b_2}}{\sqrt{1 + \frac{a_1^2}{b_1^2}}\sqrt{1 + \frac{a_2^2}{b_2^2}}}$$

which is equivalent to

$$\cos\theta = \frac{\frac{a_1a_2 + b_1b_2}{b_1b_2}}{\sqrt{\frac{a_1^2 + b_1^2}{b_1^2}}\sqrt{\frac{a_2^2 + b_2^2}{b_2^2}}} = \frac{\frac{a_1a_2 + b_1b_2}{b_1b_2}}{\frac{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}{b_1b_2}} ,$$

or

$$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}} . \tag{6}$$

Since $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$, Equation (6) is always defined. Furthermore, Equation (6) is valid even when L_1 or L_2 is vertical. We shall leave the justification as an exercise.

When two lines intersect, two pairs of vertical angles are formed. If the lines are not perpendicular, two of the angles are acute, while the other two are obtuse and supplementary to the acute angles. The cosine of an acute angle θ is positive, while its obtuse supplement $\angle\theta'$ is such that $\cos\theta' = -\cos\theta$. Thus, if we wish to obtain only the acute or right angle between lines L_1 and L_2, we consider

$$\cos\theta = \frac{|a_1a_2 + b_1b_2|}{\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}} \,. \tag{7}$$

Example 4. Find the measure of the acute angle between $L_1 = \{(x,y) : 2x - 7y + 25 = 0\}$ and $L_2 = \{(x,y) : 3x - 2y - 5 = 0\}$.

Solution.

$$\cos\theta = \frac{|2\cdot 3 + (-7)(-2)|}{\sqrt{2^2 + (-7)^2}\cdot\sqrt{3^2 + (-2)^2}} = \frac{20\sqrt{53\cdot 13}}{53\cdot 13} \approx .762 \,,$$

and $\qquad \theta \approx 40^\circ$.

Example 5. Let (ℓ_1, m_1) and (ℓ_2, m_2) be pairs of direction numbers for lines L_1 and L_2 respectively. Show that L_1 is perpendicular to L_2 if and only if $\ell_1\ell_2 + m_1m_2 = 0$.

Solution. This suggests a special case of Equation (6),

$$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}} \,,$$

where a_1, b_1 and a_2, b_2 are the coefficients in general forms of equations for L_1 and L_2 respectively. We are considering perpendicularity, which is equivalent to $\cos \theta = 0$ or the condition

$$a_1a_2 + b_1b_2 = 0 .$$

We have already observed that $(-b, a)$ are direction numbers for a line $L = \{(x,y) : ax + by + c = 0 , \text{ where } a^2 + b^2 \neq 0\}$. This is true in general, as we shall ask you to justify in the exercises. Thus, we may write $a_1 = k_1m_1$, $b_1 = -k_1\ell_1$, $a_2 = k_2m_2$, and $b_2 = -k_2\ell_2$, where k_1 and k_2 are constants such that $k_1{}^2 + k_2{}^2 \neq 0$. We substitute these in the necessary and sufficient condition above to obtain

$$k_1m_1 \cdot k_2m_2 + (-k_1\ell_1)(-k_2\ell_2) = 0 ,$$

which is equivalent to

$$\ell_1\ell_2 + m_1m_2 = 0 . \tag{8}$$

Since the three equations are equivalent, both the statement and its converse follow.

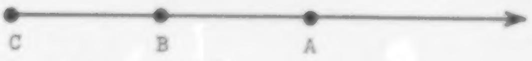
Exercises 2-7

1. Show that the relationship described by the Law of Cosines

$$\big(d(A,B)\big)^2 = \big(d(A,C)\big)^2 + \big(d(B,C)\big)^2 - 2d(A,C)d(B,C) \cos C$$

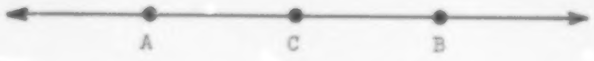
is also valid in the cases illustrated by

(a)



and

(b)



That is, justify the use of the Law of Cosines with angles of measure 0° and 180° .

2. Show that Equation (6) in the text is valid when

   (a) one line is vertical. (Let $L_1 = \{(x,y) : a_1x + c_1 = 0 , a_1 \neq 0\}$ and $L_2 = \{(x,y) : a_2x + b_2y + c_2 = 0 , a_2^2 + b_2^2 \neq 0\}$)

   (b) both lines are vertical. (Let $L_1 = \{(x,y) : a_1x + c_1 = 0 , a_1 \neq 0\}$ and $L_2 = \{(x,y) : a_2x + c_2 = 0 , a_2 \neq 0\}$.)

3. Which, if any, of the lines with the given equations are parallel? perpendicular? the same line?

   $L_1 : 3x - 4y = 12$

   $L_2 : y = \frac{4}{3}x - 3$

   $L_3 : 8x + 6y - 15 = 0$

   $L_4 : \frac{x}{4} - \frac{y}{3} = 1$

   $L_5 : \frac{x - 3}{-6 - 3} = \frac{y - 1}{-11 - 1}$

4. Find an angle between each of the pairs of lines with the given equations.

   (a) $2x - 3y + 1 = 0$, $x - 2y + 3 = 0$
   (b) $x + 2y + 3 = 0$, $y = 2x - 4$
   (c) $y = 3$, $x + y = 7$
   (d) $3x + 2y + 5 = 0$, $x - 2y + 5 = 0$
   (e) $y = 2x - 5$, $4x - 2y + 7 = 0$
   (f) $y = 2$, $x = 3$

5. If $P = (a,b)$, $Q = (-b,a)$, and $a^2 + b^2 \neq 0$, show that $\overleftrightarrow{OP} \perp \overleftrightarrow{OQ}$.

6. Let $L_1 = \{(x,y) : 2x - 3y + 4 = 0\}$ and $L_2 = \{(x,y) : 3x + y - 2 = 0\}$. Write an equation in general form of a line L_3 which is:

   (a) $\parallel L_1$ and contains the origin.
   (b) $\parallel L_2$ and contains the point $(1,5)$.
   (c) $\perp L_1$ and contains the point $(3,4)$.
   (d) $\perp L_2$ and contains the point $(2,-1)$.

7. Find an equation for a line meeting the following conditions:

   (a) Parallel to $L = \{(x,y) : 2x - 5y + 7 = 0\}$ and containing $P_1 = (2,7)$
   (b) Perpendicular to $L = \{(x,y) : 3x + 2y - 1 = 0\}$, containing $(2,7)$.
   (c) The perpendicular bisector of $\overline{AB}$, if $A = (-3,2)$ and $B = (5,-1)$.
   (d) Parallel to the x-axis and containing $P_1 = (5,7)$.
   (e) Parallel to the y-axis and containing $P_1 = (5,7)$.

8. Quadrilateral ABCD is a parallelogram. Find the coordinates of D if $A = (1,2)$, $B = (5,7)$, $C = (8,-3)$. If the order of the vertices of the parallelogram were not specified, how many possibilities would there be for D ?

9. A line L_1 makes an angle whose cosine is $\frac{3}{10}\sqrt{10}$ with $L_2 = \{(x,y) : 3x - y + 5 = 0\}$. What is the slope of L_1 ? Find its equation if it contains the point $(1,-2)$.

10. Let $A = (5,1)$, $B = (-2,3)$, and $C = (-3,4)$.
    - (a) Write the equations of $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CA}$ in general form.
    - (b) What is the slope of each of these lines?
    - (c) Find the measures of the three angles of triangle ABC.
    - (d) Write equations of the lines containing the altitudes of triangle ABC in general form.

11. Let $L_1 = \{(x,y) : a_1x + b_1y + c_1 = 0 \text{, where } a_1^2 + b_1^2 \neq 0\}$ and
$L_2 = \{(x,y) : a_2x + b_2y + c_2 = 0 \text{, where } a_2^2 + b_2^2 \neq 0\}$.
Let L_1' be perpendicular to L_1 and contain the origin and let L_2' be perpendicular to L_2 and contain the origin.
    - (a) Write equations for L_1' and L_2' in general form.
    - (b) If L_1 and L_2 form an $\angle\theta$, prove that there is an $\angle\phi$, formed by L_1' and L_2', such that $\cos\phi = \cos\theta$.
    - (c) Interpret the results of Part (b) in words.

12. Show that if lines L_1 and L_2 have pairs of direction cosines (λ_1,μ_1) and (λ_2,μ_2) respectively, then
    - (a) $\lambda_1\lambda_2 + \mu_1\mu_2 = \cos\theta$, where $\angle\theta$ is an angle formed by L_1 and L_2,
    - (b) $|\lambda_1\lambda_2 + \mu_1\mu_2| = \cos\theta$, where $\angle\theta$ is the least angle formed by L_1 and L_2, and
    - (c) $\lambda_1\lambda_2 + \mu_1\mu_2 = 0$ if and only if L_1 and L_2 are perpendicular.

2-8. Normal and Polar Forms of an Equation of a Line.

In this section we shall introduce forms of an equation of a line which display the geometric properties discussed in the last section. We shall also consider a related expression for the distance between a point and a line.

Normal Form. The results of Example 5 in Section 2-6 suggest another characterization of a line in a plane. This characterization leads to yet another form of an equation of a line; the form has several useful applications.

Once a rectangular coordinate system has been defined in a plane, any directed segment $\overrightarrow{OP}$, emanating from the origin and terminating at another point P in the plane, is determined by the distance $d(O,P)$ and the direction cosines, $\cos\alpha = \lambda$ and $\cos\beta = \mu$, of the ray $\overrightarrow{OP}$. In the plane any line L which does not contain the origin may be described simply as the set of points which is perpendicular, or normal, to the directed segment $\overrightarrow{OP}$ at P. The directed segment $\overrightarrow{OP}$ is also said to be normal to L, and is called the normal segment of L. The distance $d(O,P)$ is called the normal distance of L (and is, of course, the distance from O to L).

In Figure 2-16 we let $\overrightarrow{OP_0}$ be the normal segment of L and let $p = d(O,P_0)$. Then $P_0 = (p\cos\alpha, p\cos\beta) = (p\lambda, p\mu)$. Now $(p\lambda, p\mu)$ is also a pair of direction numbers for the line $\overleftrightarrow{OP_0}$. If $p = (x,y)$ is any point of L other than P_0, $(x - p\lambda, y - p\mu)$ is a pair of direction numbers for L.

As we have seen in Example 5 of Section 2-7, L is normal to $\overleftrightarrow{OP_0}$ at P_0 if and only if $p\lambda(x - p\lambda) + p\mu(y - p\mu) = 0$. We note that the coordinates of the point P_0 also satisfy this equation.

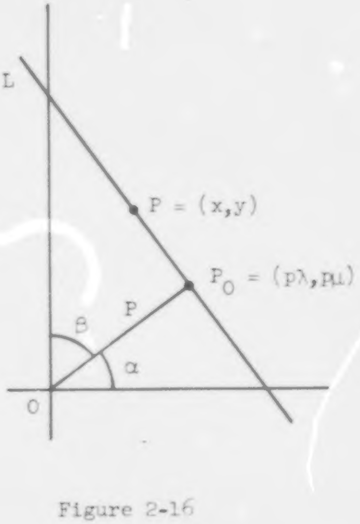


Figure 2-16

The equation is equivalent to

$$\lambda x + \mu y - p(\lambda^2 + \mu^2) = 0 .$$

Since $\lambda^2 + \mu^2 = 1$, this may be written as

(1) $$\lambda x + \mu y - p = 0 ,$$

which is called a normal form of an equation of a line. We cannot stress too strongly that in this form λ and μ are not direction cosines of the line itself, but of the normal segment. The constant p is always positive and is the distance between the origin and the line.

We may always express an equation of a line in general form; Example 5 in Section 2-6 also suggests how we may find the normal form of an equation of a line L which does not contain the origin. Let $L = \{(x,y) : ax + by + c = 0,$ where $(a^2 + b^2)c \neq 0\}$. The normal form of such an equation is a special case of the general form. Both are linear equations, and two linear equations are equivalent if and only if their corresponding coefficients are proportional. Thus, the pair (a,b) is equivalent to the normalized pair (λ,μ) of direction numbers for the normal segment. Consequently, (a,b) is a pair of direction numbers for the normal segment and

$$(\lambda,\mu) = \left(\frac{a}{\sqrt{a^2 + b^2}} , \frac{b}{\sqrt{a^2 + b^2}}\right) \text{ or } \left(\frac{-a}{\sqrt{a^2 + b^2}} , \frac{-b}{\sqrt{a^2 + b^2}}\right).$$

Our choice between these two possibilities is determined by the requirement that $p > 0$. If $c < 0$ in the equation $ax + by + c = 0$, we divide by $\sqrt{a^2 + b^2}$ to obtain the normal form; if $c > 0$, we divide by $-\sqrt{a^2 + b^2}$.

Example 1. Write $3x - 4y + 12 = 0$ in normal form.

Solution. Since the constant term is positive, we divide by $-\sqrt{3^2 + (-4)^2} = -5$ to obtain

$$-\frac{3}{5}x + \frac{4}{5}y - \frac{12}{5} = 0 .$$

We see from the equation that the normal distance is $\frac{12}{5}$, $\cos \alpha = -\frac{3}{5}$, and $\cos \beta = \frac{4}{5}$.

Example 2. Put the equation $-6x - 5y - 20 = 0$ in normal form.

Solution.
$$-\frac{6}{\sqrt{61}}x - \frac{5}{\sqrt{61}}y - \frac{20}{\sqrt{61}} = 0 .$$

We have not considered lines containing the origin. In the general form of an equation for such a line L, c is zero. There is no directed segment normal to the line emanating from the origin, nor is there a unique standard procedure in this case. Some mathematicians hold that there are two normal forms corresponding to the normal rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as illustrated in Figure 2-17; others prefer a unique form corresponding to the normal ray for which $0 \le \alpha < 180^\circ$ and $0 \le \beta \le 90^\circ$. In the first case we obtain a normal form by dividing a general form with $c = 0$ by either $\sqrt{a^2 + b^2}$ or $-\sqrt{a^2 + b^2}$; in the second case, we obtain a unique normal form by dividing by $\sqrt{a^2 + b^2}$ when $b > 0$, by $-\sqrt{a^2 + b^2}$ when $b < 0$, and by a when $b = 0$. You may follow either convention.

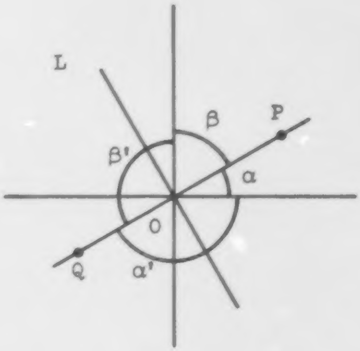


Figure 2-17

Example 3. Find the normal forms of equations of the lines

(a) $L_1 = \{(x,y) : 3x + 4y = 0\}$.

(b) $L_2 = \{(x,y) : 3x - 2y = 0\}$.

(c) $L_3 = \{(x,y) : -2x = 0\}$.

Solution.

(a) Alternate forms: $\frac{3}{5}x + \frac{4}{5}y = 0$ or $-\frac{3}{5}x - \frac{4}{5}y = 0$

Unique form: $\frac{3}{5}x + \frac{4}{5}y = 0$.

(b) Alternate forms: $\frac{3}{\sqrt{13}}x - \frac{2}{\sqrt{13}}y = 0$ or $-\frac{3}{\sqrt{13}}x + \frac{2}{\sqrt{13}}y = 0$

Unique form: $-\frac{3}{\sqrt{13}}x + \frac{2}{\sqrt{13}}y = 0$

(c) Alternate forms: $x = 0$ or $-x = 0$

Unique form: $x = 0$.

A useful application related to the normal form is to find the distance between a point $P_1 = (x_1,y_1)$ and a line $L = \{(x,y) : \lambda x + \mu y - p = 0\}$.

We illustrate this situation in Figure 2-18. F is the projection of P_1 onto L and we wish to find $d(P_1,F)$. There exists a unique line L_1 which is parallel to L and which contains P_1. L_1 is represented by the equation $\lambda x + \mu y - p_1 = 0$. Since L_1 contains (x_1,y_1), $\lambda x_1 + \mu y_1 - p_1 = 0$ or $p_1 = \lambda x_1 + \mu y_1$.

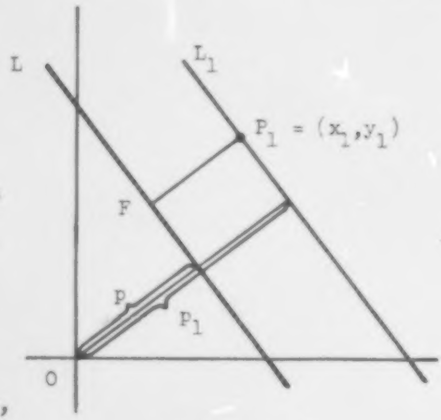


Figure 2-18

There are several cases to consider, including the following two:

i) O and P_1 are on opposite sides of L as in Figure 2-18. In this case, $d(P_1,F) = p_1 - p = \lambda x_1 + \mu y_1 - p$.

ii) P_1 is on the same side of L as O; P_1 is farther than O from L. In this case, the normal segment to L_1 has the opposite sense of direction and its direction cosines are $-\lambda, -\mu$. Hence, its normal distance is $-\lambda x_1 - \mu y_1$, or $-p_1$, and $d(P_1,F) = p + (-p_1) = |\lambda x_1 + \mu y_1 - p|$.

You may find it helpful to draw a figure to illustrate the second situation.

We leave the other possibilities as an exercise. In each case the distance d between the point $P_1 = (x_1,y_1)$ and the line $L = \{(x,y) : \lambda x + \mu y - p = 0\}$ is given by

$$d = |\lambda x_1 + \mu y_1 - p| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}. \tag{2}$$

Example 3. Find the distance between $P = (3,-10)$ and $L = \{(x,y) : 3x - 4y + 12 = 0\}$.

Solution. From Equation (2) we obtain

$$d = \frac{|3(3) - 4(-10) + 12|}{\sqrt{3^2 + (-4)^2}} = \frac{61}{5} = 12.2.$$

Polar Form. The analytic representation of a line in a plane with a polar coordinate system is similar to the normal form.

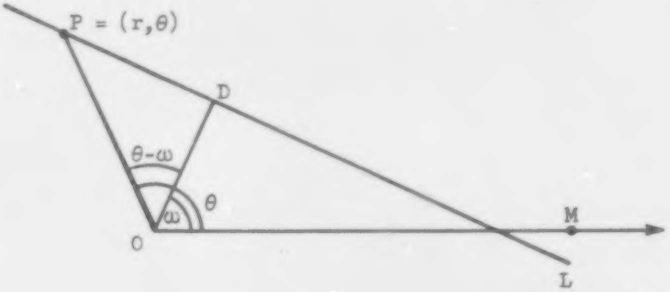


Figure 2-19

In Figure 2-19 we illustrate a line L in a plane with a polar coordinate system. Let $\overrightarrow{OD}$ be the normal segment to L, let p be the normal distance, and let $\angle\omega$ be the polar angle of D. If $P = (r,\theta)$ is any point of L other than D, then in right triangle ODP we have

$$r\cos(\theta - \omega) = p, \tag{3}$$

which is called the polar form of an equation of a line which does not contain the pole. We note that $D = (p,\omega)$ satisfies Equation (3) and that, since $\cos(\omega - \theta) = \cos(\theta - \omega)$, the equation is valid for points whose polar angle has measure θ which is less than ω.

Points are on a line L containing the pole if and only if they may all be described by the same or equivalent polar angles. Thus, the representations of a line containing the pole are

$$L = \{(r,\theta) : \theta = k + n\pi,\ \text{where } k \text{ is real and } n \text{ is an integer}\},$$

or

$$L = \{(r,\theta) : \theta = k + 180n^\circ,\ \text{where } k \text{ is real and } n \text{ is an integer}\}.$$

The appearance of the degree symbol in the second representation does not mean that the right-hand member of the equation does not represent a simple real number; rather, it is a convention to indicate that the angle is measured in degrees.

Example 4.

(a) Find a polar form of an equation of the line with inclination 135° and whose distance from the pole is 2.

(b) Find a polar equation for a line containing the pole with inclination 60°.

Solution.

(a) If the line intersects the polar axis, the polar angle of the normal segment is $\frac{\pi}{4}$, and the polar form of an equation is

$$r \cos(\theta - \tfrac{\pi}{4}) = 2 .$$

If the line intersects the ray opposite to the polar axis, the polar angle of the normal segment is $\frac{5\pi}{4}$, and the polar form of an equation is

$$r \cos(\theta - \tfrac{5\pi}{4}) = 2 .$$

(b) The line has polar equations

$$\theta = \frac{\pi}{3} + n\pi \text{ , where } n \text{ is an integer,}$$

or

$$\theta = 60^\circ + 180n^\circ \text{ , where } n \text{ is an integer.}$$

If a line has already been represented in a rectangular coordinate system as

$$L = \{(x,y) : ax + by + c = 0 , \quad a^2 + b^2 \neq 0\} ,$$

we may obtain a polar equation in the related polar coordinate system simply by substitution from the relations $x = r \cos \theta$ and $y = r \sin \theta$. The equation becomes

$$(4) \qquad a\, r \cos \theta + b\, r \sin \theta + c = 0 \text{ , where } a^2 + b^2 \neq 0 .$$

In order to see how this equation is related to the usual polar form, we recall that $ax + by + c = 0$ has the equivalent normal form $\lambda x + \mu y - p = 0$, with the corresponding coefficients proportional. Furthermore, $\lambda = \cos \alpha$ and $\mu = \cos \beta$, where $\angle\alpha$ and $\angle\beta$ are the direction angles of the normal segment. In the polar coordinate system which we have assumed to relate the coordinates, we let $\angle\omega$ be a polar angle which contains the normal segment of L. Thus $\omega = \pm \alpha$ and $\cos \omega = \cos \alpha = \lambda$. Furthermore,

$\sin\omega = \cos\beta = \mu$. If you have worked Exercise 7 of Section 2-6, you should already be aware that this is true; otherwise, you should justify now that it is so.

Let $\lambda x + \mu y - p = 0$ be the normal form of Equation (4). We substitute for λ , μ , x , and y to obtain

$$\cos\omega \cdot r\cos\theta + \sin\omega \cdot r\sin\theta - p = 0 ,$$

or

$$r(\cos\theta\cos\omega + \sin\theta\sin\omega) = p ,$$

which is equivalent to

$$r\cos(\theta - \omega) = p .$$

Example 5. Assume the usual orientation of the polar axis and find the polar form of an equation of the line

(a) 2 units to the right of the pole and perpendicular to the polar axis,

(b) 3 units above the pole and parallel to the polar axis,

(c) 1 unit to the left of the pole and perpendicular to the polar axis,

(d) 4 units below the pole and parallel to the polar axis.

(e) $L = \{(x,y) : x + \sqrt{3}y - 12 = 0\}$.

Solution.

(a) Since the length and polar angle of the normal segment are 2 and 0 respectively, the polar form of an equation is $r\cos\theta = 2$.

(b) $r\cos(\theta - \frac{\pi}{2}) = 3$. A simpler equation is $r\sin\theta = 3$.

(c) $r\cos(\theta - \pi) = 1$. Another equation is $r\cos\theta = -1$.

(d) $r\cos(\theta - 270^\circ) = 4$. Another equation is $r\sin\theta = -4$.

(e) $x + \sqrt{3}y - 12 = 0$ is equivalent to the normal form

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y - 6 = 0 ,$$

and the corresponding polar equation

$$\frac{1}{2} r\cos\theta + \frac{\sqrt{3}}{2} r\sin\theta - 6 = 0 ,$$

or

$$r\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) = 6 . \qquad (5)$$

If we let $\frac{1}{2} = \cos\omega$ and $\frac{\sqrt{3}}{2} = \sin\omega$, we obtain $\frac{\pi}{3}$ as a suitable value for ω. We substitute in Equation (5) to obtain

$$r\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right) = 6 ,$$

$$r\cos\left(\frac{\pi}{3} - \theta\right) = 6 ,$$

or

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 6 ,$$

which is in polar form.

Example 6. Assume the usual relationship between the polar axis and the x- and y-axes and write an equivalent equation in rectangular coordinates for

$$r\cos(\theta - \omega) = p .$$

Solution. If we expand $\cos(\theta - \omega)$, we obtain the equation

$$r\cos\theta\cos\omega + r\sin\theta\sin\omega = p .$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, this is equivalent to

$$x\cos\omega + y\sin\omega = p . \qquad (6)$$

Because $\cos\omega = \lambda$ and $\sin\omega = \mu$, Equation (6) is sometimes called the normal form of an equation of a line.

Exercises 2-8

1. Write each of the following equations in normal form:

   (a) $4x - 3y + 15 = 0$
   (b) $5x + 12y - 65 = 0$
   (c) $3x - 2y - 6 = 0$
   (d) $5y - 3x + 12 = 0$
   (e) $y = 3x - 7$
   (f) $y = -\frac{8}{15}x + 2$
   (g) $12x - 5y = 0$
   (h) $7y = 20$
   (i) $9x + 15 = 0$
   (j) $\frac{x}{12} - \frac{y}{5} = 1$
   (k) $\frac{y}{8} - \frac{x}{15} = 1$
   (l) $y - 2 = \frac{3}{4}(x - 5)$

2. For Parts (a) and (b) of Exercise 1, draw the normal segment by using the information concerning α , β , and p which is supplied by the equation. Then draw the line perpendicular to the normal segment at its terminal point. Verify that this is the line represented by the given equation.

3. Without using rectangular coordinates write in polar form the equation of a line

   (a) which is parallel to the polar axis and 4 units above it.

   (b) which is perpendicular to the polar axis and 4 units to the right of the pole.

   (c) through the pole with slope $\sqrt{3}$.

   (d) which contains the point $(-3,135^{\circ})$ and has inclination 45° .

   (e) which contains the point $(3,0)$ and has inclination 30° .

   (f) which contains the point $(2,\frac{\pi}{4})$ and has inclination 45° .

   (g) which is perpendicular to the line with equation $r\cos(\theta - \frac{\pi}{3}) = 2$ and contains the point $(4,\frac{\pi}{2})$.

   (h) which is parallel to the line with equation $r\cos(\theta - \frac{\pi}{4}) = 1$ and contains the point $(2,-135^{\circ})$.

4. Transform each of the following equations to polar form.

   (a) $x - 4 = 0$
   (b) $y + 4 = 0$
   (c) $x = 0$
   (d) $x + y + 2 = 0$
   (e) $3x - 2y + 6 = 0$
   (f) $x + \sqrt{3}y - 2 = 0$
   (g) $15y - 8x + 34 = 0$

5. Let $L = \{(x,y) : \lambda x + \mu y - p = 0 \text{ , where } \lambda^2 + \mu^2 = 1\}$ and let $P_1 = (x_1,y_1)$. Show that the distance between P_1 and L is $|\lambda x_1 + \mu y_1 - p|$ when

   (a) P_1 is on L .

   (b) P_1 is on the same side of L as the origin 0 ; P_1 is closer than 0 to L .

   (c) P_1 is on the same side of L as 0 ; P_1 and 0 are equidistant from L .

6. Find the distance between P and L :

   (a) $P = (6,8)$; $L = \{(x,y) : 12x - 5y + 26 = 0\}$.

   (b) $P = (-3,2)$; $L = \{(x,y) : 3x - 4y - 5 = 0\}$.

   (c) $P = (-5,-7)$; $L = \{(x,y) : y = 4x - 7\}$.

   (d) $P = (4,-5)$; $L = \{(x,y) : \frac{x}{7} + \frac{y}{5} = 1\}$.

   (e) $P = (8,11)$; $L = \{(x,y) : y - 4 = \frac{7}{5}(x - 3)\}$.

7. Find equations of the lines bisecting the angles formed by the lines $L_1 = \{(x,y) : 3x - 4y + 5 = 0\}$ and $L_2 = \{(x,y) : 12x + 5y - 13 = 0\}$.

   Hint: How is an angle bisector described as a locus?

8. Find equations of the lines bisecting the angles formed by $L_1 = \{(x,y) : 3x - 4y + 12 = 0\}$ and $L_2 = \{(x,y) : 12x - 5y - 60 = 0\}$.

   (See Exercise 7.)

9. Find equations of the lines bisecting the angles formed by the lines $L_1 = \{(x,y) : \lambda_1 x + \mu_1 y - p_1 = 0 , \lambda_1{}^2 + \mu_1{}^2 = 1\}$ and $L_2 = \{(x,y) : \lambda_2 x + \mu_2 y - p_2 = 0 , \lambda_2{}^2 + \mu_2{}^2 = 1\}$.

   (See Exercise 7.)

10. Write the equation $r \cos \theta - 3 = 0$ in rectangular coordinates.

11. Write the equation $x - y = 0$ in polar coordinates.

12. Write the equation $x^2 + y^2 = 36$ in polar coordinates.

13. Write the equation $r = 4 \cos \theta$ in rectangular coordinates.

    Hint: Multiply both members of the equation by r . Check that the pole is in the graph of the original equation. Explain why you must make this check.

14. Write the equation $r = 2a \cos \theta$ in rectangular coordinates.

    (See Exercise 13.)

15. Transform to rectangular form.

    (a) $\theta = 60^\circ$

    (b) $r \sin \theta + 4 = 0$

    (c) $r = 5$

16. Sketch the locus of each equation in Exercise 15.

17. (a) Transform $x^2 + y^2 - 4x = 0$ into polar coordinates.

    (b) Transform $r = 5 \cos \theta - 3 \sin \theta$ into rectangular coordinates.

    (c) Transform $r \cos(\theta - \frac{3\pi}{2}) = 4$ into rectangular coordinates.

    (d) Transform $(x^2 + y^2 + y)^2 = x^2 + y^2$ into polar coordinates.

2-9. Summary.

In this chapter you have encountered many topics which were already familiar from various sources. Our hope is that by gathering them together, we have offered you not only the chance to refresh your memory, but also new insight into the coherence and application of these ideas.

We first considered the basis for coordinates on a line and the characterization of subsets of a line in terms of coordinates. Next we reviewed with care the rectangular coordinate system in the plane and various analytic representations of a line in the plane.

Polar coordinates may well be a concept new to you. Relations of both mathematical interest and physical importance may often be represented most simply by equations in polar coordinates.

We have stressed our freedom of choice in introducing coordinate systems. The ease of our solution of problems depends in part upon our foresight in establishing a framework of reference.

In problem solving the danger always exists that we might let the algebra do our thinking for us. A geometric interpretation will both guide and control our application of algebraic techniques. Throughout this chapter we have emphasized the roles of algebra and geometry in the interpretation of such concepts as congruence, betweenness, direction on a line, the measure of angles, and the measure of distance between points and lines.

In the next chapter we shall study vectors. Vectors form in themselves a bridge between geometry and algebra, for they are geometric objects for which algebraic operations are defined.

Review Exercises - Section 2-6 through Section 2-8

1. Find a pair of direction numbers, a pair of direction cosines, and a pair of direction angles for

   (a) the line containing the points $(-3,7)$ and $(4,-3)$.

   (b) a line with slope $\frac{24}{25}$.

   (c) a ray emanating from $(2,3)$ and containing $(-4,8)$.

   (d) the line $L = \{(x,y) : 6x - 7y + 4 = 0\}$.

   (e) $L = \{(x,y) : \frac{x - 2}{5 - 2} = \frac{y + 4}{-7 + 4}\}$.

   (f) $L = \{(x,y) : y = \frac{7}{4}x + 9\}$.

   (g) $L = \{(x,y) : \frac{x}{5} + \frac{y}{-10} = 1\}$.

   (h) $L = \{(x,y) : y + 2 = \frac{-1 + 2}{3 - 5}(x - 5)\}$.

2. In each part below determine whether the three points are collinear.

   (a) $(11,13)$, $(-4,1)$, and $(1,5)$.
   (b) $(1,-2)$, $(-5,7)$, and $(6,-12)$.
   (c) $(23,17)$, $(-1,-1)$, and $(-17,-13)$.
   (d) $(0,-4)$, $(-3,8)$, and $(5,-11)$.

In Exercises 3-8 let $A = (-3,1)$, $B = (2,5)$, $C = (4,-1)$.

3. Find the distances: $d(A,B)$, $d(A,C)$, $d(B,C)$.

4. Write in general form the equations of the three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$.

5. Use the results of Exercise 4 to find the lengths of the three altitudes of $\triangle ABC$.

6. Use the results of Exercises 3 and 5 to find the area of $\triangle ABC$.

7. In $\triangle ABC$, find equations of

   (a) the line containing the bisector of $\angle A$.
   (b) the line containing the bisector of $\angle B$.
   (c) the line containing the bisector of $\angle C$.

In Exercises 8-11 , let $L_1 = \{(x,y) : 2x - 3y + 6 = 0\}$,
$L_2 = \{(x,y) : 3x + 4y - 12 = 0\}$,
and $L_3 = \{(x,y) : x - 2y = 0\}$.

8. Find the distance from

(a) A to each of the lines L_1 , L_2 , L_3 .

(b) B to each of the lines L_1 , L_2 , L_3 .

(c) C to each of the lines L_1 , L_2 , L_3 .

9. Find equations for the two angle bisectors of the angles formed by

(a) L_1 , L_2 .

(b) L_1 , L_3 .

(c) L_2 , L_3 .

10. Find the distances between the parallel lines:

(a) L_1 as above, and $L_4 = \{(x,y) : 2x - 3y + 12 = 0\}$.

(b) L_2 as above, and $L_5 = \{(x,y) : 3x + 4y - 1 = 0\}$.

(c) L_3 as above, and $L_6 = \{(x,y) : x - 2y + 10 = 0\}$.

11. Find two points on L_1 which are 5 units away from L_2 .

12. Find the angles between $L_1 = \{(x,y) : \frac{x - 2}{5 - 2} = \frac{y - 4}{3 - 4}\}$

and $L_2 = \{(x,y) : \frac{x - 3}{4 - 3} = \frac{y - 2}{4 - 2}\}$.

13. Show that $L_1 = \{(x,y) : \frac{x - 3}{-2 - 3} = \frac{y - 2}{5 - 2}\}$ is perpendicular to

$L_2 = \{(x,y) : \frac{x - 1}{4 - 1} = \frac{y - 4}{5 - 4}\}$.

14. Find the angles between L_1 and L_2 , where L_1 contains the points (3,4) and (-1,-1) , and L_2 contains the points (-4,6) and (3,0) .

15. Find the measure of the angle whose sides have pairs of direction cosines, $\left(-\frac{2}{\sqrt{5}} , \frac{1}{\sqrt{5}}\right)$ and $\left(\frac{7}{\sqrt{58}} , \frac{3}{\sqrt{58}}\right)$ respectively.

16. Show that triangle ABC is a right triangle, where A = (3,4) , B = (-2,7) , and C = (6,9) .

17. Find the normal form of the equations

(a) $3x - 7y + 29 = 0$.

(b) $y = \frac{20}{21}x + 58$.

(c) $\frac{x}{\text{[illegible]}} + \frac{y}{\text{[illegible]}} = 1$.

(d) $3x - 7y = 0$.

(e) $7 = 5x$.

18. Find the polar form of the equation of the line

(a) which intersects the polar axis at $(2,0)$ and has inclination $\frac{5\pi}{6}$.

(b) which is perpendicular to the polar axis at a point 4 units from the pole on the ray opposite to the polar axis.

(c) contains the pole and the point $(7,147^\circ)$.

19. Transform to rectangular coordinates:

(a) $r \cos(\theta - \frac{7\pi}{6}) = 5$

(b) $3r \sin \theta - 4r \cos \theta = 12$

20. Transform to polar coordinates:

(a) $\frac{x}{7} + \frac{y}{8} = 1$

(b) $y = \frac{8}{15}x - 12$

Challenge Exercises

For each of Exercises 1-6 write an equation to represent all lines,

1. parallel to $3x - 4y + 10$ [illegible] ,
2. perpendicular to $3x - 4y + 10 = 0$,
3. containing the origin,
4. containing the point $(2,3)$,
5. containing the point $(4,0)$ and parallel to line in Exercise 1 ,
6. having slope -3 .
7. Prove analytically that the lines containing the bisectors of the angles formed by any two intersecting lines are perpendicular.

8. Prove: If $P_1 = (x_1,y_1)$ is not on $L = \{(x,y) : ax + by + c = f(x,y) = 0\}$, then $f(x,y) = f(x_1,y_1)$ is an equation of a line parallel to L.

In Exercises 9-13 let $A = (0,0)$, $B = (1,0)$, and $C = (a,b)$, where $b \neq 0$.

9. Prove that the lines containing the altitudes of triangles ABC are concurrent at a point H. Find the coordinates of H.

10. Prove that the lines containing the medians of triangle ABC are concurrent at a point G. Find the coordinates of G.

11. Prove that the lines containing the bisectors of the angles of triangle ABC are concurrent at a point I. Find the coordinates of point I.

12. Prove that the perpendicular bisectors of the sides of triangle ABC are concurrent at a point E. Find the coordinates of point E.

13. Prove that the points H, G, and E in Exercises 9, 10, and 12 collinear. Find an equation of the line containing them.

Chapter 3

VECTORS AND THEIR APPLICATIONS

3-1. Why Study "Vectors"?

The use of vectors is becoming increasingly important. For example, many of the problems regarding space travel and ordinary air travel on the earth are solved by vector methods.

Vectors were created by the mathematical physicists William R. Hamilton and Herman Grassman in about the middle of the nineteenth century to solve the many problems involving forces and motion. Since that time vectors have been applied in many branches of science, engineering, and mathematics. The work of Hamilton and Grassman was based on the earlier development of analytic geometry by René Descartes and Pierre Fermat in the seventeenth century.

Vector methods and the non-vector methods of analytic geometry are both widely used in proving geometric theorems and they have become so interwoven that it is at times impossible to separate them. In fact, several books have been published recently under titles such as "Analytic Geometry: A Vector Approach", and courses in calculus make extensive use of both vector and non-vector methods interchangeably. This is one of the principal reasons for including this chapter in our book--to give you an additional tool to apply to find interesting relations among geometric objects and to prove some geometric theorems. An additional reason is the future need in scientific or engineering studies or in mathematics courses.

To understand what follows you should recall what you learned in your course in geometry. If you have studied about vectors before, part of this material will serve as a review and you may be interested in comparing the two approaches to the subject. However, no knowledge of vectors is assumed.

3-2. Directed Line Segments and Vectors.

In Chapter 2 we encountered directed line segments, which possess both direction and magnitude. A simple example of this geometric concept is that of a motion or displacement along a line. Let us say a boy starts at a given point and walks two miles. We don't know much about his trip until we are told the direction in which he walks or the point at which he ends. A displacement can then be represented in one of two ways:

(a) By a directed segment extending a given distance in a given direction from a given point.

(b) By a pair of points, one identified as the starting or initial point, the other as the ending or terminal point.

The symbol $\overrightarrow{AB}$ is used to denote such a directed line segment whose initial point is A and whose terminal point is B.

DEFINITION. By the magnitude of the directed line segment $\overrightarrow{AB}$ we mean $d(A,B)$, the length of the associated segment $\overline{AB}$.

We now turn our attention to the concept of a vector, which is closely related to the geometric concept of a directed line segment. Vectors were created by physicists to deal with concepts such as force, acceleration, velocity, flow of heat, and flow of electricity.

To understand this new concept, we need the following definition:

DEFINITION. Directed line segments will be considered equivalent if and only if they

(1) lie on the same or parallel lines,

(2) have the same sense of direction, and

(3) have the same magnitude.

For convenience, we shall use the term "parallel" in the sense of statement (1). The phrase "if and only if" means that the statement and its converse are both true.

DEFINITION. The infinite set of directed line segments equivalent to any given directed line segment is called a vector.

To understand more fully the concept of a vector let us recall an analogy from arithmetic. Here we have an infinite set of equivalent fractions which represent the same quantity; e.g. $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{5}{10}, \frac{11}{22}, \ldots\}$. Such a set is called a rational number.

It is common in many texts to use the word *vector* to mean, not the whole set of equivalent directed line segments, but any single member of that set. When convenient, and when there is no ambiguity we will follow this procedure. When we use the word *vector* in this way, and say that two *vectors are equal*, we mean they are members of the same set of equivalent directed line segments. In the case of the representation of rational numbers, when we say $\frac{2}{4} = \frac{3}{6}$ we mean that these two fractions represent the same rational number. We shall represent a vector by any of its members and we shall denote such directed line segments by $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, ...

Each rational number has a representative which is considered the "simplest", and that is the member whose numerator and denominator have no common factor. In the example above, $\frac{1}{2}$ is the simplest representative of the rational number.

In the same way, it will be convenient to have a "simplest" representative for each vector. For this purpose we require a reference point in space which we shall call the *origin*. Any point in space can serve as the origin, and to emphasize this freedom, we state the following principle:

> *ORIGIN PRINCIPLE*: Vectors may be related to any point in space as an origin.

The usefulness of this principle will become evident when vectors are applied to the solution of problems.

After an origin is selected in space, each vector (or equivalent set of directed line segments) contains a unique member with this origin as its initial point. We shall call this member the *origin-vector* and it will serve as the "simplest" representative of the vector. The symbol $\overrightarrow{A}$ will be the origin-vector representation for the vector $\overrightarrow{a}$, $\overrightarrow{B}$ for $\overrightarrow{b}$, ... as shown in Figure 3-1. Note that to each point A of the plane there now corresponds a unique origin-vector $\overrightarrow{A}$.

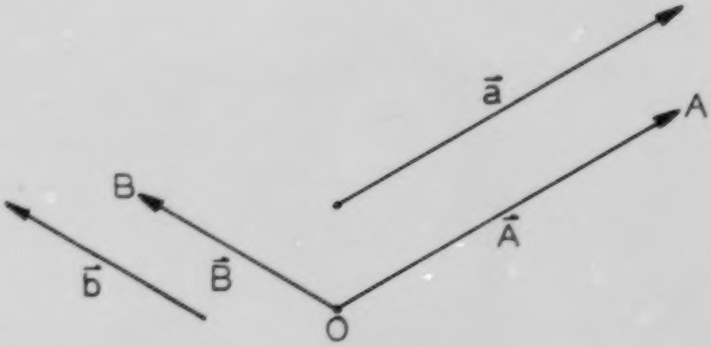


Figure 3-1

It is important to note that we do not always wish to use the simplest representative. For example, in adding $\frac{1}{2}$ and $\frac{1}{3}$, we find it most convenient to use the member $\frac{3}{6}$ instead of $\frac{1}{2}$ and $\frac{2}{6}$ instead of $\frac{1}{3}$. Likewise, in dealing with vectors, we shall frequently find it more convenient to use a representative of its set other than the origin-vector.

Vectors are very frequently associated with real numbers. In discussions involving vectors, real numbers will be referred to as *scalars*. The scalar which is the length of $\vec{a}$ will be denoted by $|\vec{a}|$ and will be referred to as its *magnitude* or *absolute value*. Other examples of scalars are the measures of angle, area, mass, and temperature. You will find it helpful to compare these with the examples of vectors given earlier.

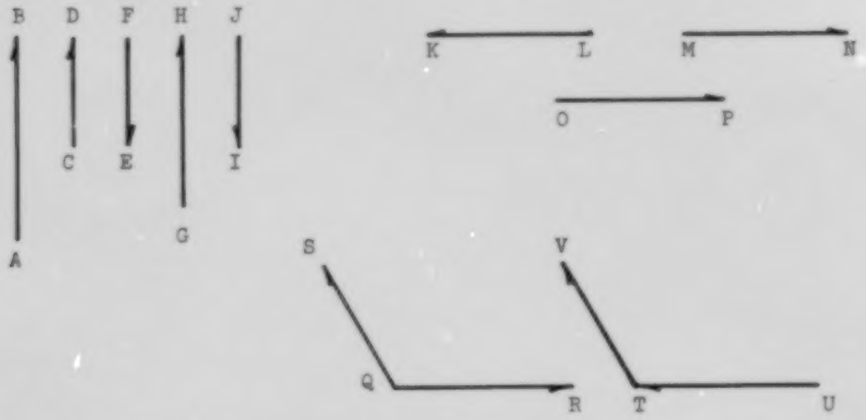
> *DEFINITIONS*. Any origin corresponds to an object called the *zero vector* and is denoted by $\vec{0}$.
>
> A vector of unit length is called a *unit vector*. Note that $\frac{a}{|a|}$ is the unit vector along $\vec{a}$.

Note also that the zero vector has zero magnitude but no particular direction. A unit vector exists in every direction.

Exercises 3-2

1. Draw a vector from (3,2) as defined in this chapter and indicate its simplest representative.

2. For the figures below indicate the sets of equivalent directed line segments.

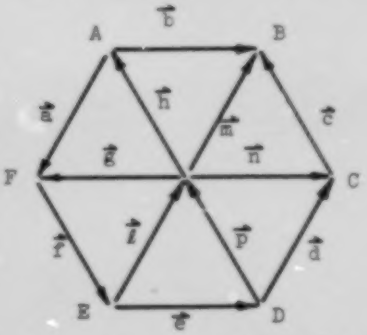

B
D
F
H
J
C
E
I
G
A
K
L
M
N
O
P
S
V
Q
R
T
U

3. Given the vertices A , B , C , and D of a parallelogram. List all the directed line segments determined by ordered pairs of these points. Which belong to the same vector?

4. Figure ABCDEF is a regular hexagon. In the diagram, find three replacements for $\overrightarrow{x}$ and $\overrightarrow{y}$ to make each of these statements true:

   (a) $\overrightarrow{x} = \overrightarrow{y}$

   (b) $\overrightarrow{x} = -\overrightarrow{y}$


A
$\overrightarrow{b}$
B
$\overrightarrow{a}$
$\overrightarrow{h}$
$\overrightarrow{m}$
$\overrightarrow{c}$
$\overrightarrow{g}$
$\overrightarrow{n}$
F
C
$\overrightarrow{f}$
$\overrightarrow{l}$
$\overrightarrow{p}$
$\overrightarrow{d}$
E
$\overrightarrow{e}$
D

5. Show the simplest representatives of four different unit vectors on a plane with a rectangular coordinate system. Do the same on a plane with a polar coordinate system.

6. List five geometric or physical concepts not listed in this section, which can be represented by vectors.

3-3. Sum and Difference of Vectors. Scalar Multiplication.

To get anything of either mathematical interest or physical usefulness, it is necessary to introduce operations on vectors. Since forces are conveniently represented by vectors, we may consider the problem of replacing two forces acting at a point by a single force called the resultant. A Dutch scientist, Simon Stevin (1548-1620) experimented with this problem and discovered that the resultant force could be represented by the diagonal of a parallelogram whose sides represented the original forces.

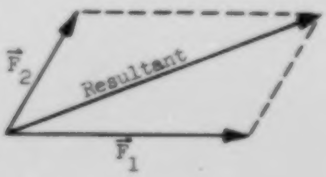


Figure 3-2

Thus a definition of vector addition is made which is consistent with observations of the physical world.

Before presenting such a definition, there is an important distinction to make between the use of origin-vectors and other vectors. You must be aware of this distinction.

We have already agreed in the statement of the "Origin Principle" that vectors may be related to any point in space as an origin. One reason for stating this principle is that it is more convenient to deal with origin-vectors when we seek a geometric interpretation.

We are about to define operations with vectors and prove several theorems. In order that the use of origin-vectors will not limit the application of the results we state the following principle:

ORIGIN-VECTOR PRINCIPLE. The sum and difference of vectors and the product of a vector by a scalar is equivalent to the sum, difference, and scalar product of their respective origin vectors.

There is one more significant statement to make in this regard. All proofs using origin-vectors depend in part upon the fact that all such vectors have a common initial point. The extension of such proofs to vectors in general can readily be made by choosing for any vectors those representatives which have a common initial point.

In other words, the algebraic relationships between vectors will hold in general, but the geometric interpretation must be limited to the geometric conditions assumed in the development.

DEFINITION.

(1) Let $\vec{P}$ and $\vec{Q}$ be two non-zero vectors not lying in the same line and with a common initial point O. We define the vector sum of $\vec{P}$ and $\vec{Q}$, designated by $\vec{P} + \vec{Q}$, to be the unique vector with initial point O and whose terminal point is the vertex opposite O in the parallelogram formed with $\vec{P}$ and $\vec{Q}$ as sides.

(2) If $\vec{P}$ and $\vec{Q}$ have the same direction, $\vec{P} + \vec{Q}$ is the vector with the same direction, and with magnitude equal to the sum of the magnitudes of $\vec{P}$ and $\vec{Q}$. If $\vec{P}$ and $\vec{Q}$ have opposite directions, $\vec{P} + \vec{Q}$ is the vector with the same direction as the vector of larger magnitude, and with magnitude equal to the absolute value of the difference of the two magnitudes.

(3) For any vector $\vec{P}$, $\vec{P} + \vec{0} = \vec{0} + \vec{P} = \vec{P}$, where $\vec{0}$ denotes the zero vector.

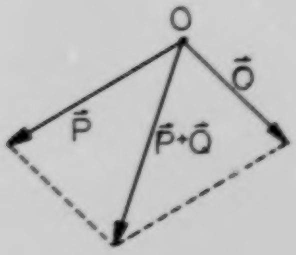


Figure 3-3

In arithmetic one usually considers multiplication as repeated addition of the same number. For example, $3 \times 2 = 2 + 2 + 2$. An analogous definition is made for the multiplication of a vector by a scalar. Thus $3\vec{A} = \vec{A} + \vec{A} + \vec{A}$. The second part of the above definition also tells us that $\vec{A} + \vec{A} + \vec{A}$ is a vector parallel to A, with the same sense of direction, and a magnitude three times as large. Generalizing this idea, one can state the following definition:

DEFINITION. Let r be a real number and $\vec{P}$ any vector. Then $r\vec{P}$ is defined by

(1) If $r > 0$, then $r\vec{P}$ is the vector with same direction as $\vec{P}$ and magnitude r times the magnitude of $\vec{P}$.

(2) If $r < 0$, then $r\vec{P}$ is the vector with direction opposite to $\vec{P}$ and magnitude $|r|$ times the magnitude of $\vec{P}$.

(3) If $r = 0$, then $r\vec{P} = \vec{0}$.

(4) If $r = 1$, then $r\vec{P} = \vec{P}$.

When $r = -1$, $r\vec{P} = (-1)\vec{P}$ and we denote this vector by the symbol $-\vec{P}$. The vector $-P$ has the opposite sense of direction of P but has the same magnitude as shown in Figure 3-4.

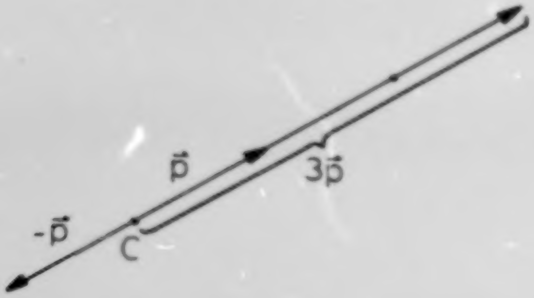


Figure 3-4

In accordance with our earlier definitions, we note that if $r \neq 0$, $r\vec{P}$ is always parallel to $\vec{P}$.

It is now possible to define one kind of division of two vectors.

DEFINITION. $\frac{\vec{A}}{\vec{B}} = k$, a scalar, if and only if $\vec{A} = k\vec{B}$; that is, if $\vec{A}$ and $\vec{B}$ are parallel.

We now can also make the following definition:

DEFINITION. $\vec{A} - \vec{B}$ means $\vec{A} + (-\vec{B})$. The quantity $\vec{A} - \vec{B}$ is called the *difference* of the two vectors $\vec{A}$ and $\vec{B}$.

Thus, in order to find the difference of two vectors, $\vec{A}$ and $\vec{B}$, we merely need to add the negative of the second to the first as shown in Figure 3-5.

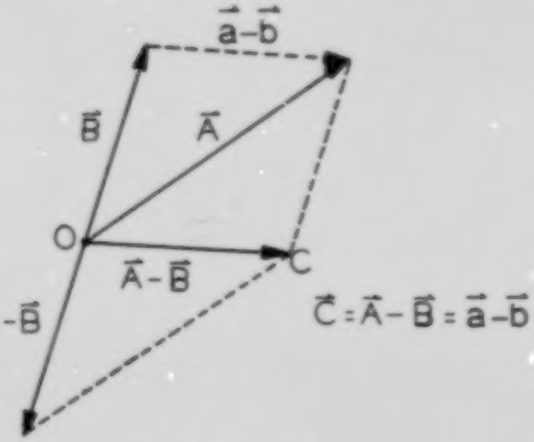


Figure 3-5

Figure 3-5 also shows that if $\vec{A} - \vec{B} = \vec{C}$, then $\vec{A} = \vec{B} + \vec{C}$.

Now that we have made the above definitions we are in a position to illustrate the distinction between the use of origin-vectors and other vectors referred to on p. 98. For example, the sum of vectors $\vec{a}$ and $\vec{b}$ in Figure 3-6 is equivalent to the sum of their respective origin-vectors $\vec{A}$ and $\vec{B}$.

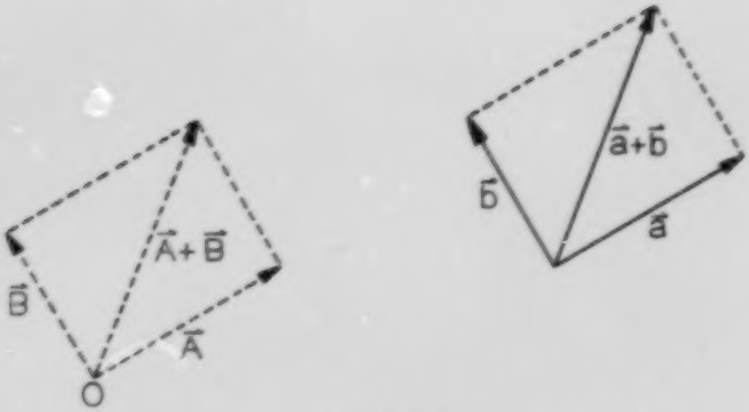

$\vec{A}+\vec{B}$
$\vec{B}$
$\vec{A}$
O
$\vec{a}+\vec{b}$
$\vec{b}$
$\vec{a}$

Figure 3-6

It is not even necessary that vectors $\vec{a}$ and $\vec{b}$ have the same initial point. (See Figure 3-7)

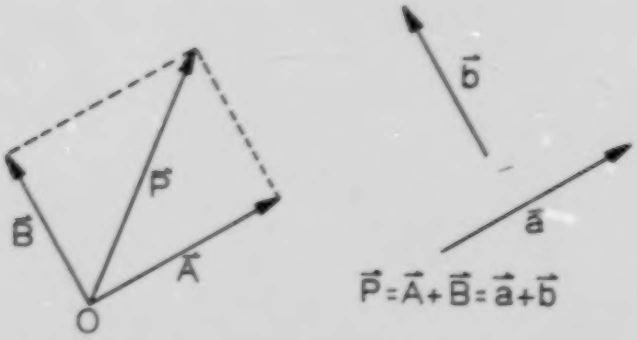

$\vec{P}$
$\vec{B}$
$\vec{A}$
O
$\vec{b}$
$\vec{a}$
$\vec{P}=\vec{A}+\vec{B}=\vec{a}+\vec{b}$

Figure 3-7

An important application of the above principle is shown in Figure 3-8 where the sum of $\vec{a}$ and $\vec{b}$ can be found by considering the equivalent of $\vec{b}$ with its initial point coincident with the terminal point of $\vec{a}$. This method can be applied to three or more vectors.

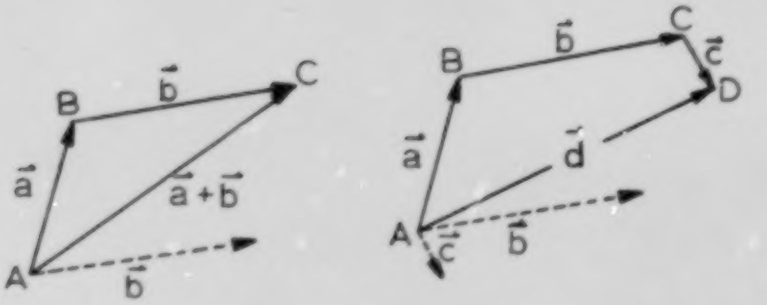

B
$\vec{b}$
C
$\vec{a}$
$\vec{a}+\vec{b}$
A
$\vec{b}$
C
$\vec{b}$
B
$\vec{c}$
D
$\vec{a}$
$\vec{d}$
A
$\vec{c}$
$\vec{b}$

Figure 3-8

In applying vector methods, physicists and other scientists often consider that they "move around a diagram", and then equate the corresponding vector sums. We could "move" from A to D directly, or from A through B and C to D. If the vector from A to D is called $\vec{d}$, then $\vec{d} = \vec{a} + \vec{b} + \vec{c}$. Likewise, one can go from A to C via two routes with the result that $\vec{a} + \vec{b} = \vec{d} - \vec{c}$.

Exercises 3-3

1. Using the figure as given, supply the missing vector expressions.

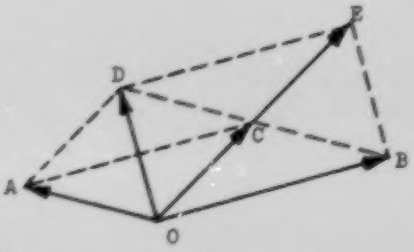
   (a) $\vec{A} + \vec{B} = ?$

   (b) $\vec{D} - \vec{A} = ?$

   (c) $\vec{A} + \vec{B} + \vec{C} = ?$

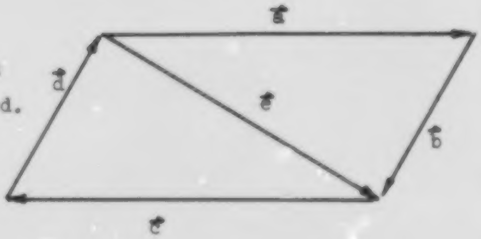
   (d) $\vec{D} + \vec{B} = r\vec{C}$ (find r)

   (e) $\vec{E} - ? = \vec{C}$



Quadrilaterals OCDA, OBCA, and OBED are parallelograms.

2. In the figure, A , B , C , and D are vertices of a parallelogram and determine the vectors indicated.



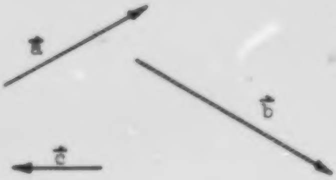
   (a) Express $\vec{c}$, $\vec{d}$, and $\vec{e}$ in terms of $\vec{a}$ and $\vec{b}$ alone.

   (b) Express $\vec{e}$ in terms of

      (i) $\vec{a}$ and $\vec{b}$
      (ii) $\vec{a}$ and $\vec{d}$
      (iii) $\vec{c}$ and $\vec{b}$
      (iv) $\vec{c}$ and $\vec{d}$

   (c) (i) What is the sum of $\vec{d}$, $\vec{e}$, and $\vec{c}$?
      (ii) What is the sum of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$?

3. Draw on paper the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as shown in the figure. Construct the vectors:



   (a) $\vec{b} + \vec{c}$
   (b) $\vec{a} - \vec{c}$
   (c) $\vec{c} - \vec{b}$
   (d) $\vec{a} + \vec{b} + \vec{c}$

4. By a drawing, show that if $\vec{a} + \vec{b} = \vec{c}$, then $\vec{b} = \vec{c} - \vec{a}$.

5. O , B , and X are collinear points. Find r such that

$$\vec{X} = r\vec{B}$$

   if

   (a) X is the midpoint of $\overline{OB}$.
   (b) B is the midpoint of $\overline{OX}$.
   (c) O is the midpoint of $\overline{BX}$.
   (d) X is $\frac{3}{4}$ of the way from O to B .
   (e) B is $\frac{2}{3}$ of the way from O to X .
   (f) O is $\frac{5}{8}$ of the way from B to X .

6. If $\vec{a} = \vec{b}$ and $\vec{c} = \vec{d}$, prove $\vec{a} + \vec{c} = \vec{b} + \vec{d}$.

7. If $|\vec{A}| = 3$, what is $|4\vec{A}|$? $|-5\vec{A}|$? $-|5\vec{A}|$?

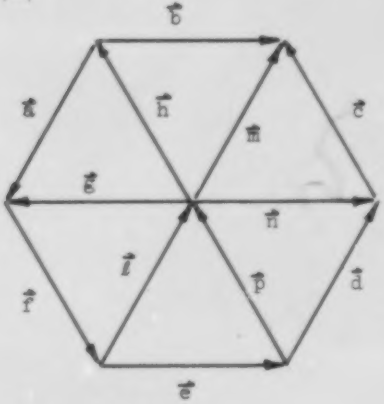
8. Prove: if $\vec{a} = \vec{b}$ and if r is a scalar, then $r\vec{a} = r\vec{b}$.

9. If $\vec{b}$ is a non-zero vector, and if $\dfrac{|\vec{a}|}{|\vec{b}|} = k$, what can you say about $|k\vec{b}|$?

10. The figure is a vector diagram based on a regular hexagon.

    (a) Write 6 vector equations which should occur to anyone in the class.

    (b) Write 6 more which are not obvious but which you could prove.



11. By using vectors, indicate 5 different paths in the plane by which one could move from $P = (1,2)$ to $Q = (4,6)$.

12. (a) If $|\vec{a}| = |\vec{b}|$, does $\vec{a} = \vec{b}$?

    (b) If $\vec{a} - \vec{b} = 0$, does $\vec{a} = \vec{b}$?

13. Prove $|\vec{a} + \vec{b}| \leqq |\vec{a}| + |\vec{b}|$

14. Letting 1 inch represent 2 miles, find graphically the resultant motion if a car travels 4 miles north and then 5 miles southeast, assuming the car travels in a plane.

15. Using the idea of resultant vectors and a scale of 1 inch to represent 2 miles per hour, solve the following problem graphically.

    A river has a 3 mile per hour current. A motor boat moves directly across the river (perpendicular to the current) at 5 miles per hour. How fast and in what direction would the boat be traveling if there were no current and the same power and heading were used in crossing the river?

16. Make a vector drawing with a scale of 1 inch to represent 10 pounds to solve the following problem.

    A body is acted on by two forces, $\vec{A}$ and $\vec{B}$, which make an angle of 70° with each other. The magnitude of $\vec{A}$ is 20 pounds and that of $\vec{B}$ is 30 pounds. What is the magnitude and direction of the resultant force?

17. Show that if $\vec{A}$ and $\vec{B}$ are distinct vectors, then $\vec{A} + (-1)\vec{B} = \vec{A} - \vec{B}$ lies on a line parallel to the line through the terminal points of $\vec{A}$ and $\vec{B}$, and similarly for $\vec{B} - \vec{A}$.

18. $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are consecutive vector sides of a quadrilateral. Prove that the figure is a parallelogram if and only if $\vec{b} + \vec{d} = \vec{0}$.

19. Prove that the sum of the six vectors drawn from the center of a regular hexagon to its vertices is the zero vector.

20. If we trace the perimeter of a polygon ABCD ... PA, and assign a vector $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \ldots, \vec{p}$ corresponding to each side as we traverse it, show that the vector sum $\vec{a} + \vec{b} + \vec{c} + \vec{d} + \ldots + \vec{p} = \vec{0}$. (It is this idea that physicists have in mind when they say, "The vector sum around a closed circuit is zero."

3-4. Properties of Vector Operations

We now derive several important algebraic properties of the operation of vector addition.

THEOREM 3-1. (Commutative Property)

$$\vec{P} + \vec{Q} = \vec{Q} + \vec{P}.$$

This follows from the definition of vector sum with the help of Figure 3-3.

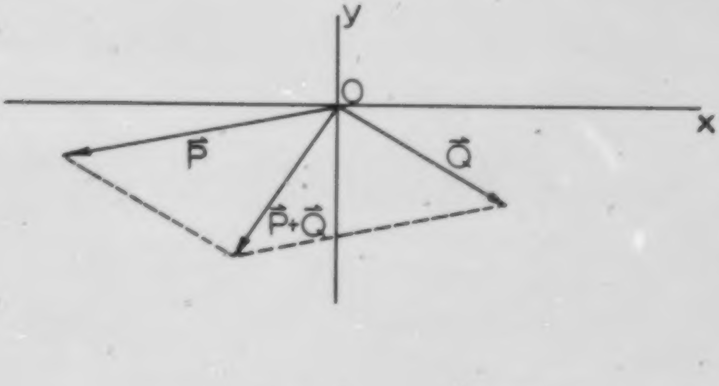


Figure 3-3

THEOREM 3-2. (Associative Property)

$$\overline{P} + (\overline{Q} + \overline{R}) = (\overline{P} + \overline{Q}) + \overline{R} .$$

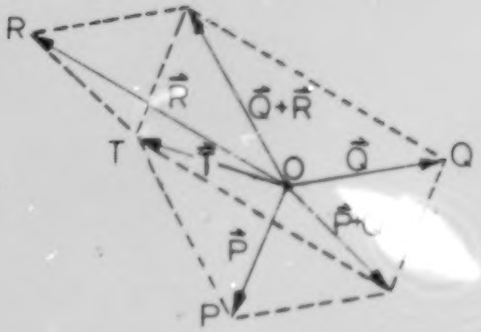


Figure 3-9

Figure 3-9 suggests a proof using the various parallelograms which appear. A much nicer proof will be given later.

THEOREM 3-3. (Additive Inverses)

For any vector $\vec{A}$, the equation

$$\vec{A} + \vec{X} = \vec{0}$$

is satisfied by $\vec{X} = (-1)\vec{A} = -\vec{A}$.

This follows immediately from the definition of addition of vectors and of $(-1)\vec{A}$.

Next we prove a theorem concerned with multiplying vectors by real numbers.

THEOREM 3-4. (Associative Property)

$$(rs)\vec{P} = r(s\vec{P}) .$$

This follows immediately from the definition of each member of the equation.

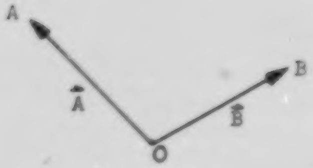
Exercises 3-4

1. By using the definition of subtraction, and the commutative and associative properties, show that

   (a) $\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$

   (b) $(\vec{A} - \vec{B}) + \vec{B} = \vec{A}$

2. Draw on paper the figure showing $\vec{A}$ and $\vec{B}$. Locate point X such that $\vec{X} = p\vec{A} + q\vec{B}$,



   (a) if $p = 1$ and $q = 1$,

   (b) if $p = \frac{1}{3}$ and $q = \frac{2}{3}$,

   (c) if $p = 0$ and $q = \frac{1}{2}$,

   (d) if $p = \frac{1}{2}$ and $q = \frac{1}{2}$,

   (e) if $p = \frac{1}{4}$ and $q = \frac{5}{4}$.

   Can you make a conjecture about the values for p and q for which X is on $\overleftrightarrow{AB}$?

3. (a) Show by a vector drawing that the subtraction of vectors, e.g., $\vec{A} - \vec{B}$, is not commutative.

   (b) Is there a relation between the two differences, i.e., does $\vec{A} - \vec{B} = r(\vec{B} - \vec{A})$?

4. Prove Theorem 3-2.

5. Show that $-(\vec{P} + \vec{Q}) = -\vec{P} - \vec{Q}$.

6. Show that $(-r)\vec{P} = r(-\vec{P})$.

3-5. Characterization of the Points on a Line.

The term "linear combination" was first mentioned in Chapter 2 in connection with finding a point of division of a line segment. Now that we know how to add and subtract vectors and how to multiply a vector by a scalar, we can combine these operations to create other vectors, such as $2\vec{A} - 3\vec{B}$, $\frac{1}{2}(\vec{B} + \vec{C})$, and $(1 - x)\vec{A} + \frac{x}{2}\vec{B}$. To formalize this idea, we state the following definition:

DEFINITION. If $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$ are n vectors and x_1 , x_2 , ..., x_n are n scalars, the vector $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$ is said to be a linear combination of $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$.

In order to use vectors to prove theorems in geometry we need several basic theorems. The first one is concerned with expressing any vector in the plane as a linear combination of other vectors in the same plane.

THEOREM 3-5. If $\vec{a}$ and $\vec{b}$ are coplanar and non-parallel, then any third vector $\vec{c}$, which lies in the plane determined by $\vec{a}$ and $\vec{b}$, can be expressed as a unique linear combination of $\vec{a}$ and $\vec{b}$.

Given: Coplanar and non-parallel vectors $\vec{a}$ and $\vec{b}$, and $\vec{c}$ lying in their plane.

Prove: $\vec{c} = x\vec{a} + y\vec{b}$ where x and y are scalars.

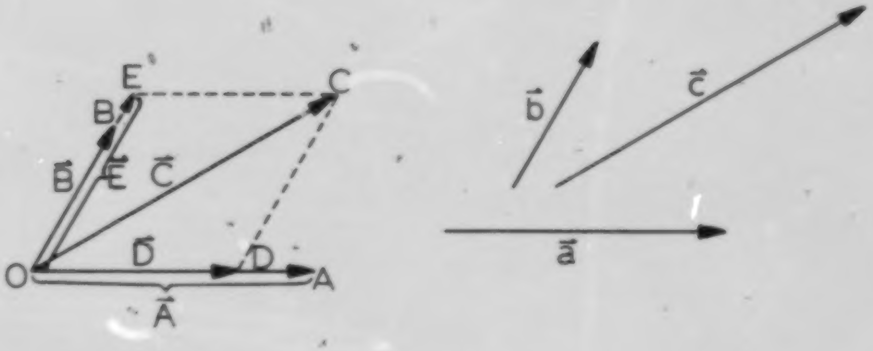


Figure 3-10

Inasmuch as vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be represented by their respective origin-vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ with terminal points A , B , and C as shown in Figure 3-10, we need only prove that $\vec{C} = x\vec{A} + y\vec{B}$. In this diagram we have chosen x and y positive although this restriction is not needed.

(1) Draw a line through C parallel to the line containing $\vec{B}$. Let D be the point of intersection of this line with the line containing $\vec{A}$.

(2) Since $\vec{D}$ is parallel to $\vec{A}$, it is some scalar multiple of $\vec{A}$. Thus, for some unique x, $\vec{D} = x\vec{A}$.

(3) Similarly, the vector $\vec{E}$, along the line containing $\vec{B}$, is a scalar multiple of $\vec{B}$. Thus, for some unique y, $\vec{E} = y\vec{B}$.

(4) Then $\vec{C} = \vec{D} + \vec{E} = x\vec{A} + y\vec{B}$ which shows $\vec{C}$ is a unique linear combination of $\vec{A}$ and $\vec{B}$. We have the equivalent statement: $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

We note that if $\vec{c}$ is parallel to $\vec{a}$ or $\vec{b}$, then $\vec{c}$ is a scalar multiple of either $\vec{a}$ or $\vec{b}$ alone.

THEOREM 3-6. (Distributive Properties)

1. $r(\vec{P} + \vec{Q}) = r\vec{P} + r\vec{Q}$.
2. $(r + s)\vec{P} = r\vec{P} + s\vec{P}$.

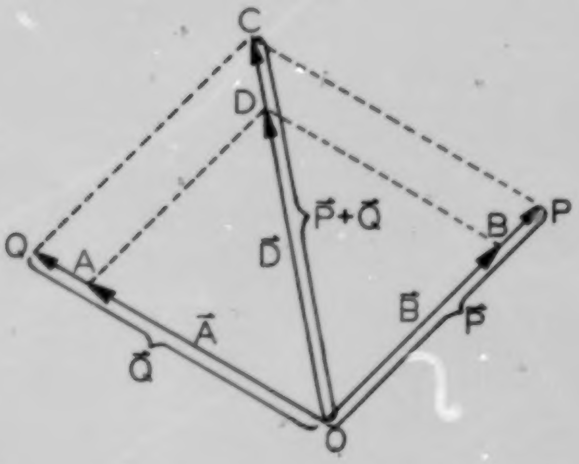


Figure 3-11

Proof of Part 1: $r(\vec{P} + \vec{Q}) = r\vec{P} + \vec{Q}$.
In this proof, we assume $\vec{P}$ and $\vec{Q}$ on distinct lines with $r > 0$.

(1) In Figure 3-11, $\vec{A} = r\vec{Q}$, $\vec{B} = r\vec{P}$.
Therefore: $|\vec{A}| = r|\vec{Q}|$, $|\vec{B}| = r|\vec{P}|$,

(2) $\dfrac{|\vec{B}|}{|\vec{A}|} = \dfrac{r|\vec{P}|}{r|\vec{Q}|} = \dfrac{|\vec{P}|}{|\vec{Q}|}$;

(3) $|\vec{B}| = d(O,B) = d(A,D)$,

$|\vec{A}| = d(O,A)$,

$|\vec{P}| = d(O,P) = d(Q,C)$,

$|\vec{Q}| = d(O,Q)$.

(4) Combining steps (2) and (3) we have $\dfrac{d(A,D)}{d(O,A)} = \dfrac{d(Q,C)}{d(O,Q)}$, and therefore $\triangle OAD \sim \triangle OQC$.

(5) $\therefore d(O,D) = r(d(O,C))$

$|\vec{D}| = r|\vec{C}| = |r\vec{C}|$.

(6) Since the vectors are in the same direction, we have $\vec{D} = r\vec{C}$.

(7) $\vec{D} = \vec{A} + \vec{B}$ or

$r\vec{C} = r\vec{Q} + r\vec{P}$, and since $\vec{C} = \vec{P} + \vec{Q}$,

$r(\vec{P} + \vec{Q}) = r\vec{P} + r\vec{Q}$.

Let us consider the special cases where the non-zero vectors $\vec{P}$ and $\vec{Q}$ are collinear. They are then parallel and have either the same or opposite senses.

If they have the same sense of direction, then

(1) By definition, $\vec{P} + \vec{Q}$ has the same sense of direction as $\vec{P}$ and $\vec{Q}$, and has magnitude $|\vec{P}| + |\vec{Q}|$.

(2) If $r > 0$, then $r(\vec{P} + \vec{Q})$ also has the same sense of direction as $\vec{P} + \vec{Q}$, $\vec{P}$, and $\vec{Q}$, and has magnitude $r(|\vec{P}| + |\vec{Q}|) = r|\vec{P}| + r|\vec{Q}|$ by definition and the distributive law.

(3) In the same way, since $r > 0$, $r\vec{P} + r\vec{Q}$ has the same sense of direction as $r\vec{P}$, $r\vec{Q}$, $\vec{P}$, and $\vec{Q}$, and has magnitude $|r\vec{P}| + |r\vec{Q}| = r|\vec{P}| + r|\vec{Q}|$.

(4) Since the vectors $r(\vec{P} + \vec{Q})$ and $r\vec{P} + r\vec{Q}$ have the same magnitude and the same sense of direction, they are equal, as was to be shown.

The case in which $\vec{P}$ and $\vec{Q}$ have opposite direction is treated in a similar fashion and the proof is left for class discussion.

The proof of the cases where $r \leq 0$ is also left for class discussion. The proof of the second part of the distributive law: $(r + s)\vec{P} = r\vec{P} + s\vec{P}$ is left as an exercise.

THEOREM 3-7. If $\vec{A}$ and $\vec{B}$ are distinct vectors not lying in the same line, then the vector $p\vec{A} + q\vec{B}$ will terminate on the line determined by the terminal points of $\vec{A}$ and $\vec{B}$ if and only if $p + q = 1$.

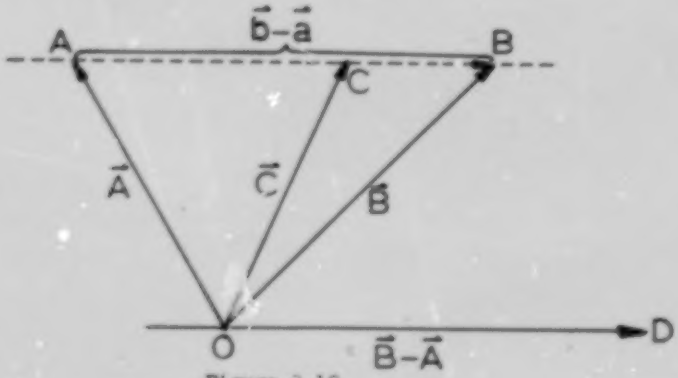


Figure 3-12

Proof:

(1) C is collinear with A and B if and only if C = A or $\overline{AC} \parallel \overline{OD}$

(2) $\overline{AC} \parallel \overline{OD}$ if and only if there exists a $q \neq 0$ such that

$$\vec{C} - \vec{A} = q(\vec{B} - \vec{A})$$

or $$\vec{C} = \vec{A} + q\vec{B} - q\vec{A}$$

or $$\vec{C} = q\vec{B} + (1 - q)\vec{A}$$

or $$\vec{C} = p\vec{A} + q\vec{B} \qquad \text{where } p + q = 1 .$$

We note that if $q = 0$, then $\vec{C} = \vec{A}$.

The statement $\vec{C} = q\vec{B} + (1 - q)\vec{A}$ is a vector form of an equation of the line through A and B.

Each particular choice of p (and consequently of q) referred to in the Theorem 3-7 determines a vector to a point on the line $\overleftrightarrow{AB}$ in Figure 3-12. We can therefore describe subsets of the line $\overleftrightarrow{AB}$ by placing conditions on the scalars p and q.

The line $\overleftrightarrow{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1\}$

The segment $\overline{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$, and $p \geq 0$, $q \geq 0\}$

The ray $\overrightarrow{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and $q \geq 0\}$

The ray $\overrightarrow{BA} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and $p \geq 0\}$

The ray opposite to $\overrightarrow{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and $q \leq 0\}$
The interior of $\overline{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and $p > 0, q > 0\}$.

Furthermore,

(i) if $\vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$, $p > 0$ and $q > 0$, then X is an interior point of $\overline{AB}$,

(ii) if $\vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and either p or q is zero, then X is an endpoint of $\overline{AB}$, and

(iii) if $\vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and either $p < 0$ or $q < 0$, then X is a point of the line exterior to $\overline{AB}$.

We observe that in the vector representation $p\vec{A} + (1 - p)\vec{B}$ the scalar is also a coordinate in one of the coordinate systems for the line. When $p = 0$, we obtain $\vec{B}$; when $p = 1$, we obtain $\vec{A}$. The value of p which determines a vector $\vec{X}$ in this vector representation of the line $\overleftrightarrow{AB}$ is also the coordinate of the point X in the coordinate system for the line with origin B and unit-point A.

THEOREM 3-8. If P divides $\overline{AB}$ in the ratio n:m, then

$$\vec{P} = \frac{m\vec{A} + n\vec{B}}{m + n}$$ where $\vec{A}$, $\vec{B}$, and $\vec{P}$ are origin-vectors to points A, B, P respectively.

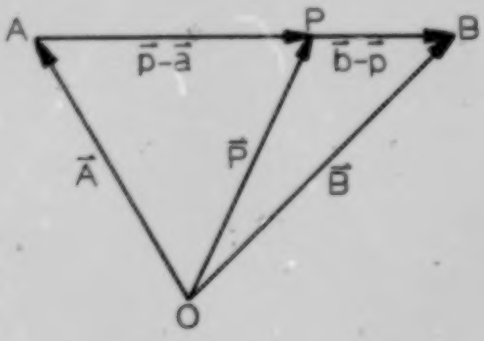


Figure 3-13

(1) Referring to Figure 3-13, $\dfrac{|\vec{p} - \vec{a}|}{|\vec{b} - \vec{p}|} = \dfrac{n}{m}$ (Given).

(2) $\dfrac{\vec{p} - \vec{a}}{\vec{b} - \vec{p}} = \dfrac{n}{m}$ (the vectors lie on the same line).

(3) $m(\vec{p} - \vec{a}) = n(\vec{b} - \vec{p})$.

(4) $m\vec{p} - m\vec{a} = n\vec{b} - n\vec{p}$.

(5) $m\vec{p} + n\vec{p} = m\vec{a} + n\vec{b}$.

(6) $(m + n)\vec{p} = m\vec{a} + n\vec{b}$, or $\vec{p} = \frac{m\vec{a} + n\vec{b}}{m + n} = \frac{m}{m + n}\vec{a} + \frac{n}{m + n}\vec{b}$.

(7) In terms of origin-vectors, we may then write:

$$\vec{P} = \frac{m\vec{A} + n\vec{B}}{m + n} = \frac{m}{m + n}\vec{A} + \frac{n}{m + n}\vec{B} .$$

Note: If P is the midpoint, then $\vec{P} = \frac{1}{2}(\vec{A} + \vec{B})$.

Exercises 3-5

1. Given vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ with their terminal points A , B , and C on a straight line, so that $\vec{C} = p\vec{A} + q\vec{B}$, $p + q = 1$.

   (a) What happens if $\vec{A}$ or $\vec{B}$ is the zero vector?

   (b) What are p and q if $\vec{C} = \vec{A}$?

   (c) What can we say about $\vec{C}$ if

      (i) $p > 0$ and $q > 0$?

      (ii) $p < 0$?

      (iii) $p = 0$?

   (d) Construct figures to illustrate the cases:

      (i) $p = q = \frac{1}{2}$

      (ii) $p = \frac{1}{3}$, $q = \frac{2}{3}$

      (iii) $p = -\frac{1}{4}$, $q = \frac{5}{4}$

      (iv) $p = \frac{3}{2}$, $q = -\frac{1}{2}$

2. (a) If the ratio of the division of a line segment is given by $n:m = 2:3$, find n and m so that $n + m = 1$.

   (b) Same as part (a) for $m:n = 5:-3$

3. Make a vector drawing to illustrate Theorem 3-5 when

   (a) $x = 2$; $y = 3$

   (b) $x = -2$, $y = 4$.

4. Prove Theorem 3-6, Part 2.

3-6. Components

We have used extensively the correspondence between points in the plane and vectors. It is fruitful to describe this correspondence in another way using the rectangular coordinates of a point. To each ordered pair of real numbers (a,b), there corresponds a unique vector emanating from O and terminating in that point and thus we make the following definition.

DEFINITION. The symbol $[a,b]$ denotes the origin-vector to point (a,b). The number a is called the x-component of the vector and the number b, the y-component of the vector.

We now describe the operations involving vectors in terms of components.

THEOREM 3-9. If $\vec{X} = [a,b]$ and $\vec{Y} = [c,d]$,

$$\vec{X} + \vec{Y} = [a + c , b + d] .$$

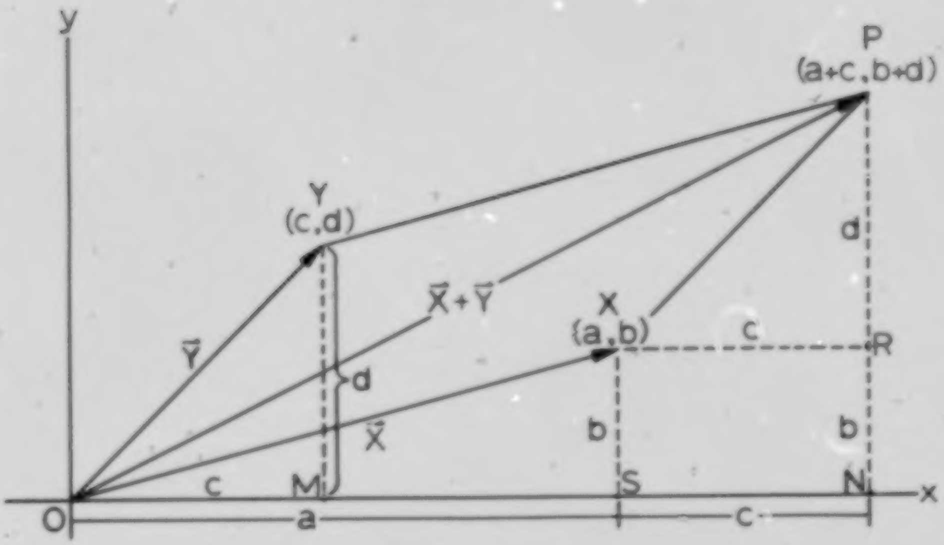


Figure 3-14

Proof. The parallelogram in Figure 3-14 is constructed according to the definition of addition of vectors.

Since $\triangle OMY \cong \triangle XRP$, $d(O,M) = d(X,R) = d(S,N) = c$ and $d(M,Y) = d(R,P) = d$. The vertex P opposite O is the point $(a + c , b + d)$, and this vertex is the terminal point of $\vec{X} + \vec{Y}$. If the vectors have the same or opposite directions, the proof follows immediately from the definition of vector addition. If $\vec{Y}$ is the zero vector $[0,0]$, then

$$[a,b] + [0,0] = X + Y = X = [a,b] = [a + 0 , b + 0] .$$

THEOREM 3-10. If $\vec{X} = [a,b]$ and r is a real number, then $r\vec{X} = [ra,rb]$.

The proof is left as an exercise.

THEOREM 3-11. We prove, using components, a theorem learned earlier: Two non-zero vectors $\vec{X}$ and $\vec{Y}$ lie in the same line through the origin, if and only if $\vec{X} = r\vec{Y}$ for some real number r .

Proof. If $\vec{Y} = [a,b]$ and $\vec{X} = [ra,rb]$, then $\vec{X}$ and $\vec{Y}$ lie in the line $ay = bx$. Conversely, if $\vec{Y} = [a,b]$ and if $\vec{X}$ lies in the line which contains $\vec{Y}$, then the components of $\vec{X}$ must satisfy the equation $ay = bx$. Hence $\vec{X} = [ra,rb]$ for some real number r .

The vector $[1,0]$ is indicated by the letter i and $[0,1]$ by j . The i and j vectors could be written as $\vec{i}$ and $\vec{j}$ but, in accordance with common usage, we shall use the simpler notation. They represent the unit vectors along the horizontal and vertical axes respectively.

If $A = (a_1,a_2)$, the origin-vector $\vec{A}$ may be written as follows:

$$\vec{A} = [a_1,a_2] = [a_1,0] + [0,a_2] = a_1[1,0] + a_2[0,1] = a_1i + a_2j .$$

Note that a_1 and a_2 are the components of $\vec{A}$; a_1i and a_2j are called the component vectors of $\vec{A}$. We observe in Figure 3-15 that any origin-vector can be written uniquely as the sum of its component vectors. The magnitude of $\vec{A}$ is $\sqrt{a_1^2 + a_2^2}$.

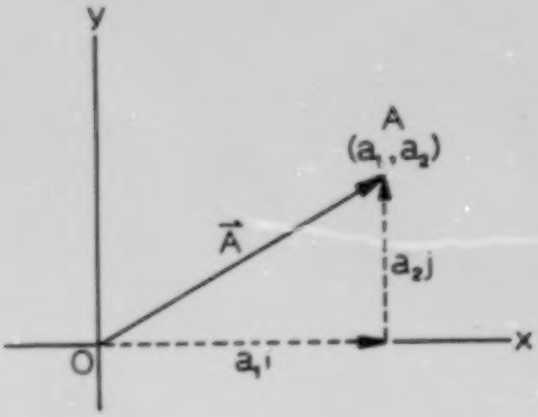


Figure 3-15

The use of components leads to a simple arithmetic of vectors, as will be seen in the following sections.

Example 1. Given $\vec{X} = [2,3]$ and $\vec{Y} = [-1,5]$,

Find $\vec{Z} = [4,-2]$ in terms of $\vec{X}$ and $\vec{Y}$.

We must find scalars r and s such that $\vec{Z} = r[2,3] + s[-1,5]$. Hence $[4,-2] = [2r,3r] + [-s,5s] = [2r - s , 3r + 5s]$.

Since the components of a given origin-vector are unique, we have:

$$2r - s = 4$$
$$3r + 5s = -2$$

We find that $r = \frac{18}{13}$, $s = \frac{-16}{13}$; hence $z = \frac{18}{13}[2,3] - \frac{16}{13}[-1,5]$.

We can form vector descriptions of lines and their subsets using components.

Example 2. Find the vector representation, in terms of a single parameter, for $\overleftrightarrow{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$.

Solution. Let $\vec{P}$ be the origin-vector to any point P on $\overleftrightarrow{AB}$.

(1) $\vec{P} = r\vec{A} + (1-r)\vec{B}$ (Theorem 3-7)

$= r[3,4] + (1 - r)[-2,3]$

$= [3r,4r] + [-2 + 2r , 3 - 3r]$

(2) Thus $\overleftrightarrow{AB} = \{P: \vec{P} = [-2 + 5r , 3 + r]\}$.

Example 3. Find, using components, a vector representation of $\overline{AB}$ wher
$A = (3,4)$ and $B = (-2,3)$.

Solution. $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$. As in Example 2, any point P
$\overline{AB}$ can be represented by

$$\overleftrightarrow{AB} = \{P : \vec{P} = r\vec{A} + (1 - r)\vec{B}\} .$$

However we must place a restriction on r so that P will lie only on $\overline{AB}$. This condition will be met if $0 \leq r \leq 1$ since $P = A$ when $r = 1$ and $P =$ when $r = 0$.

The complete solution is:

$$\overline{AB} = \{P : \vec{P} = [-2 + 5r , 3 + r] , 0 \leq r \leq 1\} .$$

Example 4. Find, using components, a vector representation of $\overrightarrow{BA}$ wher
$A = (3,4)$ and $B = (-2,3)$.

Solution. This problem differs from Example 3 in only one respect. We must now place a restriction on r so that P will lie only on $\overrightarrow{BA}$. This condition will be met if $r \geq 0$ since $P = B$ when $r = 0$ and P lies on the ray emanating from B and containing A when $r > 0$. The complete solu tion is:

$$\overrightarrow{AB} = \{P : \vec{P} = [-2 + 5r , 3 + r] , r \geq 0\} .$$

Example 5. Find the vector representation of the trisection points of $\overline{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$.

Solution. Referring to Theorem 3-8, we have

$$\vec{P} = \frac{m\vec{A} + n\vec{B}}{m + n}$$

where P divides the segment in the ratio $n:m$.

There are two points of trisection, one where $n:m = 1:2$; the other whe $n:m = 2:1$. We shall do the first part.

$$\vec{P} = \frac{2[3,4] + 1[-2,3]}{3} = \frac{2}{3}[3,4] + \frac{1}{3}[-2,3] = [\frac{4}{3}, \frac{11}{3}] .$$

Exercises 3-6

1. Find the components of
   (a) [3,2] + [4,1] .
   (b) [3,-2] + [-4,1] .
   (c) 4[5,6] .
   (d) -4[5,6] .
   (e) -1[5,6] .
   (f) - [5,6] .
   (g) 3[4,1] + 2[-1,3] .
   (h) 3[4,1] - 2[-1,3] .

2. If $\vec{A} = [3,-5]$, $\vec{B} = [-1,6]$, $\vec{C} = [2,3]$, find the components of
   (a) $2\vec{A} + 3\vec{B} - \vec{C}$.
   (b) $\vec{A} - 2\vec{B} + 3\vec{C}$.
   (c) $2(\vec{A} + \vec{B}) - 3(\vec{B} - \vec{C})$.
   (d) $5(\vec{A} - \vec{C}) + 3(\vec{C} - \vec{A})$.
   (e) $3(\vec{A} + \vec{B} - \vec{C}) + 2(\vec{A} - \vec{B} + \vec{C})$.
   (f) $5(\vec{C} - \vec{A} + \vec{B}) - 3(\vec{B} + \vec{A} - \vec{C})$.

3. What is the x component of i ? of j ?

4. Find the magnitude of the following vectors:
   (a) $i + j$.
   (b) $3i - 4j$.
   (c) $ai + bj$.
   (d) $(\cos\theta)i + (\sin\theta)j$.

5. Vector $\vec{P}$ is drawn from $A = (4,2)$ to $B = (5,-1)$. Write its origin-vector $\vec{P}$ in terms of i and j .

6. Express the zero vector $\vec{0}$ in terms of two distinct non-collinear vectors $\vec{X}$ and $\vec{Y}$ lying in the same plane.

7. In terms of i and j , describe the vector represented by the arrow extending from 0 to the midpoint of the segment joining $(2,5)$ and $(5,8)$.

8. In terms of i and j , describe
   (a) the unit vector making an angle of 30° with the x-axis.
   (b) the unit vector making an angle of -30° with the x-axis.
   (c) the unit vector having the same direction as $4i - 3j$.

9. Find x and y so that

   (a) $x[3,-1] + y[3,1] = [5,6]$.

   (b) $x[3,2] + y[2,3] = [1,2]$.

   (c) $x[3,2] + y[-2,3] = [5,6]$.

   (d) $x[3,2] + y[6,4] = [-3,-2]$ (infinitely many solutions. Why?)

10. Represent an arbitrary vector $[a,b]$ as a linear combination of

    (a) $[1,0]$ and $[0,1]$.

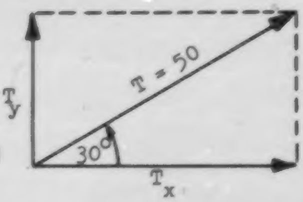
    (b) $[1,1]$ and $[-1,1]$.

    (c) $[-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}]$ and $[-1,0]$.

11. Physical forces possess both magnitude and direction and therefore may be represented by vectors. In physics problems it is often convenient to use x-components and y-components to represent the horizontal and vertical components of a force.

    Suppose a sled is being pulled along level ground by a cord making an angle of 30° with the ground. The tension (magnitude of the pulling force) in the cord is 50 pounds. What is the component of the force parallel to the ground, and what is the component of the force perpendicular to the ground?

    (Hint: With the force vector emanating from the origin, the horizontal vector will be $[T \cos 30^\circ, 0]$ and the vertical vector will be $[0, T \sin 30^\circ]$.)



12. Two forces act simultaneously at the same point. The first has a magnitude of 20 pounds, and direction 37° above the horizontal and toward the right. The other force has a magnitude of 30 pounds and direction 30° below the horizontal and toward the right. Find the vector which represents the resultant of these two forces.

13. Refer to the forces of Exercise 12.

    (a) At what angle must the second force act if the resultant acts horizontally toward the right?

    (b) At what angle must the second force act if the resultant acts vertically?

14. Suppose three forces act simultaneously at the same point. (It can be seen from the commutative and associative properties of addition for vectors that there is but one resultant for all three, no matter which two are taken first.) Find the resultant of these three forces: 20 pounds acting due west, 30 pounds acting northwest, and 40 pounds acting due south.

15. If two forces have the same magnitude but act in opposite directions, they are said to be in equilibrium and each is called the equilibrant of the other.

    (a) Find the magnitude and direction of the equilibrant of the resultant of two forces, one pulling due north with a magnitude of 20 pounds and the other pulling southeast with a magnitude of 30 pounds.

    (b) If a third force of 10 pounds acting due east is added, find the force which will provide equilibrium for the whole system.

16. A picture weighing ten pounds is suspended evenly by a wire going over a hook on the wall. If the two ends of the wire make an angle of 140° at the hook, find the tension in the wire. (See Exercise 11 for the use of "tension".)

17. Prove Theorems 3-1, 3-2, and 3-6 using components.

18. Prove Theorem 3-10.

19. Find vector representations, in terms of a single parameter for the sets described below:

    (a) $\overleftrightarrow{AB}$ where $\vec{A} = [2,3]$ and $\vec{B} = [-4,5]$
    (b) $\overleftrightarrow{AB}$ where $\vec{A} = [1,3]$ and $\vec{B} = [3,9]$
    (c) $\overleftrightarrow{AB}$ where $\vec{A} = [4,-7]$ and $\vec{B} = [4,2]$
    (d) $\overleftrightarrow{AB}$ where $\vec{A} = [2]$ and $\vec{B} = [3]$
    (e) $\overline{AB}$ where $\vec{A} = [-3,2]$ and $\vec{B} = [1,-2]$
    (f) $\overline{AB}$ where $\vec{A} = [1]$ and $\vec{B} = [2]$
    (g) $\overline{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (h) $\overrightarrow{AB}$ where $\vec{A} = [1,-2]$ and $\vec{B} = [-3,2]$
    (i) $\overrightarrow{AB}$ where $\vec{A} = [2]$ and $\vec{B} = [1]$
    (j) $\overrightarrow{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (k) $\overline{BA}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (l) $\overrightarrow{BA}$ where $\vec{A} = [1]$ and $\vec{B} = [2]$
    (m) The ray opposite to $\overrightarrow{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (n) The interior of segment $\overline{AB}$ where $\vec{A} = [-3,2]$ and $\vec{B} = [1,-2]$

20. Find the vector representations of the midpoints and trisection points of the following line segments:

    (a) $\overline{AB}$ where $A = [0,0]$ and $B = [6,12]$
    (b) $\overline{AB}$ where $A = [-3,2]$ and $B = [10,-11]$
    (c) $\overline{AB}$ where $A = [a_1,a_2]$ and $B = [b_1,b_2]$

21. Find the vector representations of the points which divide the directed segment (P,Q) in the ratio $\frac{r}{s}$ where:

    (a) $P = [4,6]$, $Q = [-1,11]$, and $\frac{r}{s} = \frac{2}{3}$

    (b) $P = [4]$, $Q = [11]$, and $\frac{r}{s} = \frac{3}{4}$

    (c) $P = [-3,-2]$, $Q = [3,2]$, and $\frac{r}{s} = 1$

    (d) $P = [-1,4]$, $Q = [9,-5]$, and $\frac{r}{s} = \frac{1}{5}$

    (e) $P = [\frac{3}{2},\frac{2}{3}]$, $Q = [\frac{1}{13},\frac{8}{13}]$, and $\frac{r}{s} = \frac{\sqrt{2}}{\pi}$

    (f) $P = [4]$, $Q = [11]$, and $\frac{r}{s} = \frac{6}{8}$

3-7. Inner Product.

Our algebra of vectors does not yet include multiplication of one vector by another. In order to make a definition which will have significant consequences, we investigate the angle between two vectors.

> **DEFINITION.** Let $\vec{X}$ and $\vec{Y}$ be any two non-zero vectors. Then by the angle between $\vec{X}$ and $\vec{Y}$ we mean the angle whose sides contain $\vec{X}$ and $\vec{Y}$. This angle has a unique degree measure between 0° and 180° (inclusive).

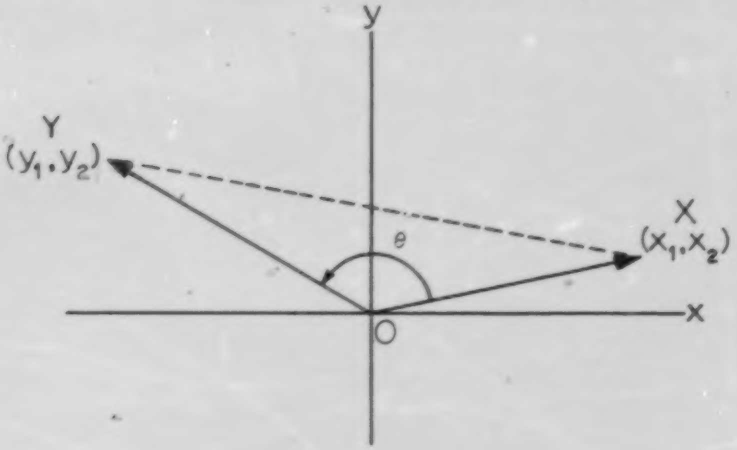


Figure 3-16

Let θ denote the angle between $\vec{X}$ and $\vec{Y}$. The law of cosines, applied to triangle OXY, enables us to write

$$\left(d(X,Y)\right)^2 = |\vec{X}|^2 + |\vec{Y}|^2 - 2|\vec{X}||\vec{Y}| \cos\theta .$$

The term $|X||Y| \cos\theta$ has significant physical applications which lead us to a useful vector concept. One such application deals with the work done in applying a force through a given distance. Since we must consider the direction and magnitude of both the force which is applied and the motion which takes place, it is customary to represent them by vectors $\vec{F}$ and $\vec{S}$, where $s = |\vec{S}|$ is the distance.

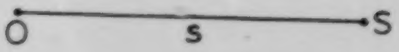


Figure 3-17

In Figure 3-17, an object at O is moved a distance s by a force $\vec{F}$. This force is applied to the object along a straight line and in the same direction as that line so that all of the force acts in the direction of motion.

On the other hand, if the force is applied at an angle θ, as shown in Figure 3-18, only that vector component of the force, $\vec{F}_x$, which produces the motion is effective in performing the work done.

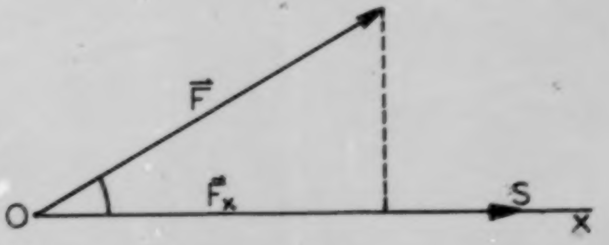


Figure 3-18

In Figure 3-18, $d(O,S) = s = |\vec{S}|$ so

$$\text{Work} = |\vec{F}_x|s = |\vec{F}|s\cos\theta = |\vec{F}||\vec{S}|\cos\theta .$$

DEFINITION. Let $\vec{X}$ and $\vec{Y}$ be any non-zero vectors. Then the inner product, $\vec{X}\cdot\vec{Y}$, of the two vectors is the real number

$$|\vec{X}||\vec{Y}|\cos\theta$$

where $|\vec{X}|$ is the magnitude of $\vec{X}$, $|\vec{Y}|$ is the magnitude of $\vec{Y}$, and θ is the angle between $\vec{X}$ and $\vec{Y}$. If either $\vec{X}$ or $\vec{Y}$ is the zero vector, $\vec{X}\cdot\vec{Y}$ is defined to be zero.

The inner product $\vec{X}\cdot\vec{Y}$ is usually read "vector X dot vector Y" and is therefore sometimes called the "dot product". Notice that the inner product is an operation that assigns to each pair of vectors a real number rather than a vector. The operation is obviously commutative.

In view of the above definition, Work = $\vec{F} \cdot \vec{S}$. Also $\vec{A} \cdot \vec{A} = |\vec{A}|^2$, $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{B}$.

Example. Evaluate $\vec{X} \cdot \vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$ and (a) $\theta = 0^\circ$, (b) $\theta = 45^\circ$, (c) $\theta = 90^\circ$, (d) $\theta = 180^\circ$.

Solution.

(a) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 0^\circ = 2 \cdot 3 \cdot 1 = 6$

(b) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 45^\circ = 2 \cdot 3 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$

(c) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 90^\circ = 2 \cdot 3 \cdot 0 = 0$

(d) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 180^\circ = 2 \cdot 3 \cdot (-1) = -6$

The inner product has many applications. One of these is a test for perpendicularity.

THEOREM 3-12. If $\vec{X}$ and $\vec{Y}$ are non-zero vectors, then they are perpendicular if and only if

$$\vec{X} \cdot \vec{Y} = 0 .$$

Proof. According to the definition of inner product

$$\vec{X} \cdot \vec{Y} = |\vec{X}| \cdot |\vec{Y}| \cos \theta .$$

This product of real numbers is zero if and only if one of its factors is zero. Since $\vec{X}$ and $\vec{Y}$ are non-zero vectors, the numbers $|\vec{X}|$ and $|\vec{Y}|$ are not zero. Therefore the product is zero if and only if $\cos \theta = 0$, which is the case if and only if $\vec{X}$ and $\vec{Y}$ are perpendicular.

The following theorem supplies a useful formula for the inner product of vectors.

THEOREM 3-13. If $\vec{X} = [x_1, x_2]$ and $\vec{Y} = [y_1, y_2]$, then

$$\vec{X} \cdot \vec{Y} = x_1 y_1 + x_2 y_2 .$$

Proof. From the law of cosines and the distance formula we can now write (see Figure 3-16)

$$\vec{X}\cdot\vec{Y} = |\vec{X}|\,|\vec{Y}|\cos\theta = \frac{|\vec{X}|^2 + |\vec{Y}|^2 - (d(X,Y))^2}{2}$$

$$= \frac{1}{2}[x_1^{\,2} + x_2^{\,2} + y_1^{\,2} + y_2^{\,2} - (x_1 - y_1)^2 - (x_2 - y_2)^2]$$

$$= \frac{1}{2}(2x_1y_1 + 2x_2y_2) = x_1y_1 + x_2y_2 \; .$$

Example 1. If $\vec{X} = [8,-6]$ and $\vec{Y} = [3,4]$, show that $\vec{X}$ and $\vec{Y}$ are perpendicular.

Solution. $\vec{X}\cdot\vec{Y} = 8\cdot 3 + (-6)\cdot 4 = 24 - 24 = 0$.
Since $\vec{X}$ and $\vec{Y}$ are non-zero vectors, Theorem 3-12 shows that they are perpendicular.

Example 2. Find the angle between the vectors $\vec{A} = [4,3]$ and $\vec{B} = [-2,2]$

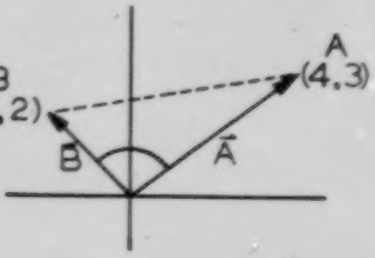


Figure 3-19

Solution.

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$$

$$\vec{A}\cdot\vec{B} = (4)(-2) + (3)(2) = -2$$

$$|\vec{A}| = 5 \; , \; |\vec{B}| = 2\sqrt{2}$$

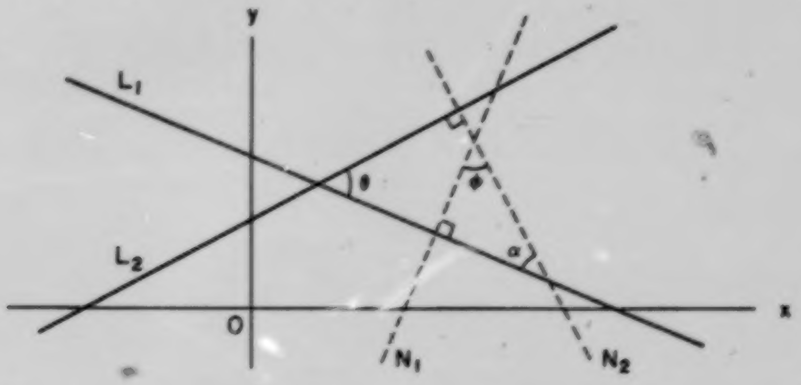
$$\therefore \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{-2}{10\sqrt{2}} = -\frac{\sqrt{2}}{10} \approx -.141$$

$$\theta \approx 98^\circ$$

We shall find further application for the formula

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} \; .$$

The Angle Between Two Lines. An application of this formula can be made to find the angles formed by two lines with equations in rectangular form. Suppose the lines are L_1 and L_2 with respective equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.



In Chapter 2 we learned that the respective normals N_1 and N_2 have direction numbers (a_1,b_1) and (a_2,b_2) . We may take these as vector components of vectors along N_1 and N_2 . From the diagram, $\angle\theta$ and $\angle\phi$ have equal measure since each is the complement of $\angle\alpha$; hence, we may find θ , the measure of the angle between L_1 and L_2 , by finding ϕ , the measure of the angle between their normals. Therefore

$$\cos\theta = \cos\phi = \frac{[a_1,b_1]\cdot[a_2,b_2]}{|[a_1,b_1]|\ \ |[a_2,b_2]|} = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\ \sqrt{a_2^2 + b_2^2}}$$

This is the same formula we found in Chapter 2 by another approach.

Example. Find the angles formed by the lines with equations $3x + 4y + 5 = 0$ and $5x + 12y + 9 = 0$.

Solution. Direction numbers for the normals to these lines are $(3,4)$ and $(5,12)$; therefore,

$$\cos\theta = \frac{[3,4]\cdot[5,12]}{|[3,4]|\,|[5,12]|} = \frac{15 + 48}{\sqrt{3^2 + 4^2}\ \sqrt{5^2 + 12^2}} = \frac{63}{5\cdot 13} = \frac{63}{65}\ ,$$

$$\cos\theta \approx .969\ , \qquad \text{and} \qquad \theta \approx 14^\circ\ .$$

The angles formed have measure 14° and 166° .

Exercises 3-7

1. If $i = [1,0]$ and $j = [0,1]$, find

   (a) $i \cdot j$

   (b) $j \cdot i$

   (c) $i \cdot i$

   (d) $j \cdot j$

   (e) $(i + j) \cdot (i - j)$

   (f) $(2i + 3j) \cdot (4i - 5j)$

   (g) $(ai + bj) \cdot (ci + dj)$.

2. If $\vec{A} = [3,-5]$, $\vec{B} = [-2,1]$, $\vec{C} = [4,-3]$, find:

   (a) $\vec{A} \cdot \vec{B}$

   (b) $2\vec{A} \cdot 3\vec{B}$

   (c) $3\vec{A} \cdot (\vec{B} + \vec{C})$

   (d) $2\vec{B} \cdot (3\vec{A} + 2\vec{C})$

   (e) $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B})$

   (f) $(2\vec{B} + 3\vec{C}) \cdot (2\vec{B} - 3\vec{C})$

   (g) $(3\vec{A} + 5\vec{B}) \cdot (3\vec{B} - 2\vec{C})$

   (h) $(\vec{A} + \vec{B} - \vec{C}) \cdot (\vec{B} - \vec{A} + \vec{C})$

   (i) $(2\vec{A} - 3\vec{B} + 4\vec{C}) \cdot (5\vec{A} - 2\vec{C} + 4\vec{B})$

   (j) $\vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{C} \cdot \vec{C}$

3. Find the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$ and $\vec{X} \cdot \vec{Y}$ is

   (a) 0

   (b) 1

   (c) -2

   (d) 3

   (e) -4

   (f) 5

   (g) 6

   (h) -6

4. Given

   (a) $\vec{A} = 4i - 3j$, find $|\vec{A}|^2$.

   (b) $\vec{B} = 12i + 5j$, find $|\vec{B}|^2$.

5. If $\vec{X} = 3i + 4j$, determine w so that $\vec{Y}$ is perpendicular to $\vec{X}$, if $\vec{Y}$ is

   (a) $wi + 4j$

   (b) $wi - 4j$

   (c) $4i + wj$

   (d) $wi - 3j$

   (e) Find an origin-vector in component form which is perpendicular to $\vec{X}$ and four times as long. (two answers)

6. Given $\vec{A} = 2i - j$ and $\vec{B} = 3i + 6j$ as sides of $\triangle AOB$; what kind of a triangle is $\triangle AOB$? Find the third side $\vec{c}$ in terms of $\vec{A}$ and $\vec{B}$. Find $\vec{C}$, the origin-vector of $\vec{c}$, in terms of its unit vectors.

7. Let $\vec{A} = 2i - 3j$, $\vec{B} = -2i + j$. Find

   (a) the angle between $\vec{A}$ and $\vec{B}$.

   (b) the work done by $\vec{A}$, considered as a force vector, in moving a particle from the origin to $S = (2,0)$ along the x-axis.

8. A sled is pulled a distance of s ft. by a force of f lbs., where F represents the force which makes an angle of θ with the horizontal. Find the work done if

   (a) $s = 100$ ft., $f = 10$ lbs., $\theta = 20^\circ$.

   (b) $s = 1000$ ft., $f = 10$ lbs., $\theta = 30^\circ$.

9. In Problem (8), how far can the sled be dragged if the number of available foot pounds of work is 1000 and if

   (a) $f = 100$ lbs., $\theta = 20^\circ$.

   (b) $f = 100$ lbs., $\theta = 89^\circ$.

10. Let $\overrightarrow{A} = (\cos \theta)i + (\sin \theta)j$ and
    $\overrightarrow{B} = (\cos \phi)i + (\sin \phi)j$.

    Draw these vectors in the xy-plane.

    (a) Find $\overrightarrow{A} \cdot \overrightarrow{B}$, $|\overrightarrow{A}|$, $|\overrightarrow{B}|$

    (b) Use those results to prove that
    $\cos(\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta$.

11. Prove: $-1 \leq \dfrac{\overrightarrow{X} \cdot \overrightarrow{Y}}{|\overrightarrow{X}||\overrightarrow{Y}|} \leq 1$.

12. Comment on the following: there is an associative law for vector addition: $(\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C})$. Therefore, there may be an associative law for inner products: $\overrightarrow{A} \cdot (\overrightarrow{B} \cdot \overrightarrow{C}) = (\overrightarrow{A} \cdot \overrightarrow{B}) \cdot \overrightarrow{C}$

3-8. Laws and Applications of the Inner (Dot) Product.

A useful fact about inner products is that they have some of the algebraic properties of products of numbers. The following theorem gives two such properties.

THEOREM 3-14. If $\overrightarrow{X}$, $\overrightarrow{Y}$, $\overrightarrow{Z}$ are any vectors, then

(a) $\overrightarrow{X} \cdot (\overrightarrow{Y} + \overrightarrow{Z}) = \overrightarrow{X} \cdot \overrightarrow{Y} + \overrightarrow{X} \cdot \overrightarrow{Z}$

(b) $(t\overrightarrow{X}) \cdot \overrightarrow{Y} = t(\overrightarrow{X} \cdot \overrightarrow{Y}) = (\overrightarrow{X}) \cdot (t\overrightarrow{Y})$.

Part (b) states "a scalar multiple of a dot product can be attached to either vector factor."

Proof. Let $\vec{X} = [x_1,x_2]$, $\vec{Y} = [y_1,y_2]$, $\vec{Z} = [z_1,z_2]$. Then

(a)
$$\begin{aligned}\vec{X}\cdot(\vec{Y}+\vec{Z}) &= [x_1,x_2]\cdot[y_1+z_1\ ,\ y_2+z_2]\\ &= x_1(y_1+z_1)+x_2(y_2+z_2)\\ &= x_1y_1+x_2y_2+x_1z_1+x_2z_2\\ &= \vec{X}\cdot\vec{Y}+\vec{X}\cdot\vec{Z}\ ,\end{aligned}$$

(b)
$$\begin{aligned}(t\vec{X})\cdot\vec{Y} &= [tx_1,tx_2]\cdot[y_1,y_2]\\ &= tx_1y_1+tx_2y_2\\ &= t(x_1y_1+x_2y_2)\\ &= t(\vec{X}\cdot\vec{Y})\ .\end{aligned}$$

Corollary. $\vec{X}\cdot(a\vec{Y}+b\vec{Z}) = a(\vec{X}\cdot\vec{Y})+b(\vec{X}\cdot\vec{Z})$.

The proofs of this corollary and the last part of Theorem 3-14 are left as exercises.

We may now use the inner product to prove theorems in geometry which involve perpendicularity.

Example 1. Show that the diagonals of a rhombus are perpendicular.

Solution. Choose the origin as one vertex of the rhombus. The two adjacent sides can be represented by the vectors $\vec{A}$ and $\vec{B}$ with $|\vec{A}| = |\vec{B}|$.

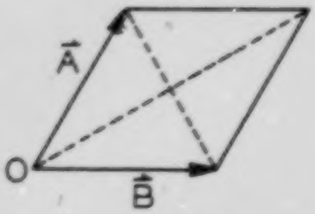


Figure 3-20

Thus one diagonal is represented by $\vec{A}+\vec{B}$ and the other diagonal is parallel to $\vec{A}-\vec{B}$. To test for perpendicularity we calculate the inner product of these two vectors, using Theorem 3-14.

$$\begin{aligned}(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) &= (\vec{A}+\vec{B})\cdot\vec{A} - (\vec{A}+\vec{B})\cdot\vec{B} \\ &= \vec{A}\cdot\vec{A} + \vec{B}\cdot\vec{A} - \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{B} \\ &= |\vec{A}|^2 - |\vec{B}|^2 .\end{aligned}$$

But $|\vec{A}| = |\vec{B}|$, so that the inner product is zero and hence the diagonals are perpendicular.

Example 2. Prove that the altitudes of a triangle are concurrent.

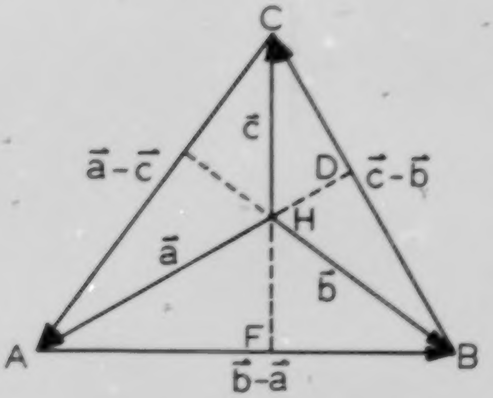


Figure 3-21

Proof. Refer to Figure 3-21: Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$. Then $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ must intersect at some point H . AH intersects BC at some point D . We must prove $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

(1) $\vec{b}\cdot(\vec{a}-\vec{c}) = \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{c} = 0$; (Why?)
thus $\vec{b}\cdot\vec{a} = \vec{b}\cdot\vec{c}$.

(2) Similarly, $\vec{c}\cdot(\vec{b}-\vec{a}) = \vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{a} = 0$;
thus $\vec{c}\cdot\vec{a} = \vec{c}\cdot\vec{b}$.

(3) $\vec{b}\cdot\vec{a} = \vec{c}\cdot\vec{a}$. (Why?)

(4) $\vec{c}\cdot\vec{a} - \vec{b}\cdot\vec{a} = 0$.

(5) $(\vec{c}-\vec{b})\cdot\vec{a} = 0$ and $\vec{a} \perp (\vec{c}-\vec{b})$.

(6) Hence $\overleftrightarrow{AD} \perp \overline{BC}$ and the three altitudes are concurrent.

The inner product can be used to derive another result. Let $\vec{X} = [x_1, x_2]$ be a non-zero vector. Then $\vec{X'} = [-x_2, x_1]$ is also a non-zero vector and we have

$$\vec{X}\cdot\vec{X'} = [x_1,x_2]\cdot[-x_2,x_1] = -x_1x_2 + x_1x_2 = 0 \, .$$

Hence by Theorem 3-12, $\vec{X}$ and $\vec{X'}$ are perpendicular and the angle between the vectors is 90° . Now let $\vec{Y} = [y_1,y_2]$ be any non-zero vector. We now calculate $\vec{X'}\cdot\vec{Y}$.

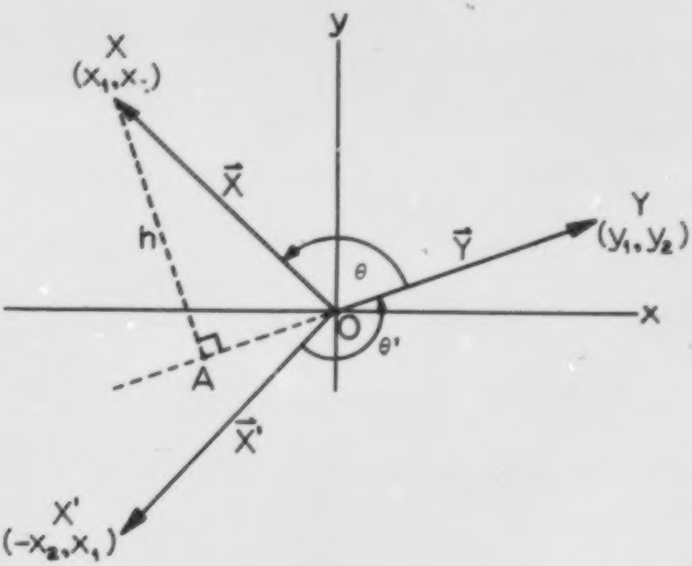


Figure 3-22

To do so we must determine the angle between the vectors $\vec{X'}$ and $\vec{Y}$. The relationship of this angle to angle θ is not always the same. In Figure 3-22 the angle θ' between $\vec{X'}$ and $\vec{Y}$ is $360^\circ - (90^\circ + \theta)$. If $\vec{Y}$ were near the positive side of the y-axis, the angle θ would be $90^\circ + \theta$. If $\vec{Y}$ were between $\vec{X}$ and $\vec{X'}$, the angle θ' would be $90^\circ - \theta$. If $\vec{Y}$ were near the negative side of the y-axis, the angle would be $\theta - 90^\circ$. Therefore, we have

$$\cos\theta' = \left\{\begin{array}{l} \cos[360^\circ - (90^\circ + \theta)] \, , \\ \cos(90^\circ + \theta) \, , \\ \cos(90^\circ - \theta) \, , \\ \text{or } \cos(\theta - 90^\circ) \, , \end{array}\right\} = \pm \sin\theta \, .$$

Therefore, in any case, since $X' = [-x_2,x_1]$,

$$\vec{X'}\cdot\vec{Y} = [-x_2,x_1]\cdot[y_1,y_2] = x_1y_2 - x_2y_1 = |\vec{X}||\vec{Y}| \cos\theta' = \pm |\vec{X}||\vec{Y}| \sin\theta \, .$$

But from the figure, we see that $|\overset{\frown}{X}| \sin\theta$ is the length of the altitude h drawn from X to line $\overleftrightarrow{OY}$ in $\triangle OXY$. Thus the area K of $\triangle OXY$ is given by

$$K = \frac{1}{2}|\overset{\frown}{Y}|h \text{ .}$$

However, since $h = |\overset{\frown}{X}| \sin\theta$,

$$K = \frac{1}{2}|\overset{\frown}{Y}|\,|\overset{\frown}{X}| \sin\theta = \frac{1}{2}|x_1y_2 - x_2y_1| \text{ .}$$

3-9. Resolution of Vectors.

In the first discussion on vector components (Section 3-6), it was noted that the vector $\overset{\frown}{X} = [a,b]$ had a as its x-component and b as its y-component.

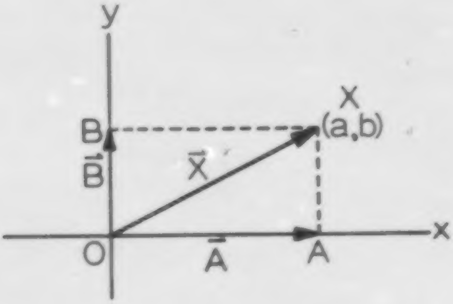


Figure 3-23

As before, we have the component vectors $ai = \overset{\frown}{A}$, and $bj = \overset{\frown}{B}$.

We now wish to extend this concept of component vectors. Consider any non-zero origin-vectors $\overset{\frown}{X}$ and $\overset{\frown}{Y}$ to points $\overset{\frown}{X}$ and $\overset{\frown}{Y}$ respectively. Let the perpendicular from X to $\overleftrightarrow{OY}$ meet $\overleftrightarrow{OY}$ in point P as indicated in Figure 3-24. Then the vectors $\overset{\frown}{m}$ and $\overset{\frown}{n}$ corresponding to $\overset{\frown}{OP}$ and $\overset{\frown}{PX}$ are called the component vectors of $\overset{\frown}{X}$ with respect to $\overset{\frown}{Y}$. This idea is not restricted to origin-vectors.

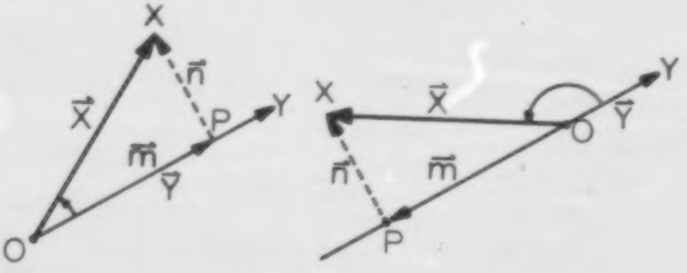


Figure 3-24

This extension of the concept of components of vectors is often helpful in physical and geometric applications, where these ideas are discussed in terms of the resolution of a vector into vector components. In the above discussion, we say that we resolve $\overrightarrow{X}$ into vector components m and n respectively parallel and perpendicular to $\overrightarrow{Y}$.

From the definition of the inner product of two vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$, we have

(1) the component of $\overrightarrow{X}$ in the direction of $\overrightarrow{Y}$,

$$\overrightarrow{X}\cos\theta = \frac{\overrightarrow{X}\cdot\overrightarrow{Y}}{|\overrightarrow{Y}|} = \overrightarrow{X}\cdot\frac{\overrightarrow{Y}}{|\overrightarrow{Y}|} \quad \text{where}$$

$\dfrac{\overrightarrow{Y}}{|\overrightarrow{Y}|}$ represents the unit vector along the $\overrightarrow{Y}$ direction.

(2) the component of $\overrightarrow{Y}$ in the direction of $\overrightarrow{X}$,

$$\overrightarrow{Y}\cos\theta = \frac{\overrightarrow{X}\cdot\overrightarrow{Y}}{|\overrightarrow{X}|} = \overrightarrow{Y}\cdot\frac{\overrightarrow{X}}{|\overrightarrow{X}|} \quad \text{where}$$

$\dfrac{\overrightarrow{X}}{|\overrightarrow{X}|}$ represents the unit vector along the $\overrightarrow{X}$ direction.

Exercises 3-8 and 3-9

1. Verify Theorem 3-14 (b) for the vectors

$$\overrightarrow{X} = [2,4]\ ,\ \overrightarrow{Y} = [-1,-3] \text{ and } t = 5\ .$$

2. If $\overrightarrow{X} = [x_1,x_2]$ and $\overrightarrow{Y} = [y_1,y_2]$, prove that $(t\overrightarrow{X})\cdot\overrightarrow{Y} = \overrightarrow{X}\cdot(t\overrightarrow{Y})$ for any scalar t.

3. Prove the corollary of Theorem 3-14.

4. (a) Supply the reasons for each step of the proof of the theorm in Example 1 following Theorem 3-14.
   (b) Same as (a) for the theorem in Example 2.

5. Find the area of the triangle determined by $\overrightarrow{A} = [3,-1]$ and $\overrightarrow{B} = [2,6]$ and check your result by any method.

6. Given $\overrightarrow{A} = 2i - 3j$ and $\overrightarrow{B} = -2i + j$. Find the component of
   (a) $\overrightarrow{A}$ upon $\overrightarrow{B}$
   (b) $\overrightarrow{B}$ upon $\overrightarrow{A}$

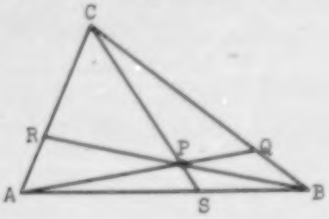
7. Given a vector representing a wind of 30 mph. from the southwest. Locate this vector in a coordinate plane where the positive side of the y-axis is considered to lie in the north direction. Resolve this vector into its $\underline{m}$ and $\underline{n}$ components (as described in Figure 3-23) with respect to:

   (a) the x and y axes.
   (b) the line $\theta = 15^\circ$.
   (c) the vector $\vec{A} = [10,15]$.

Challenge Problems

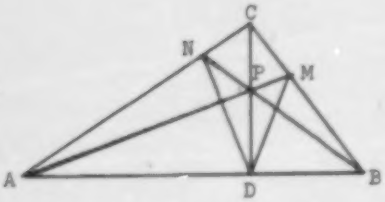
1. (Ceva's Theorem) Let P be any point not on triangle ABC . Let $\overrightarrow{AP}$, $\overrightarrow{BP}$, $\overrightarrow{CP}$ intersect $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively at Q, R, S . Show that

$$\frac{d(A,S)}{d(S,B)} \cdot \frac{d(B,Q)}{d(Q,C)} \cdot \frac{d(C,R)}{d(R,A)} = 1$$



2. In triangle ABC , let $\overline{CD} \perp \overline{AB}$ and let P be any point on $\overline{CD}$. Let $\overrightarrow{AP}$ intersect $\overline{BC}$ at M and $\overrightarrow{BP}$ intersect $\overline{AC}$ at N . Show that $\angle CDN = \angle CDM$.

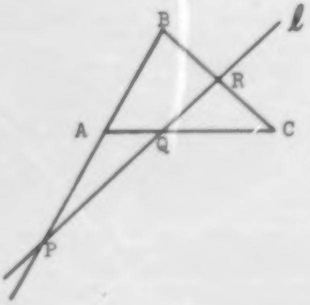
   (Hint. Take D to be 0 .)



3. (Menelaus' Theorem) Let ℓ be any line which does not pass through any vertex of triangle ABC . Let ℓ intersect $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ respectively at P, Q, R .

   Show that

$$\frac{d(A,Q)}{d(Q,C)} \cdot \frac{d(C,R)}{d(R,B)} \cdot \frac{d(B,P)}{d(P,A)} = 1 .$$



4. (a) Prove algebraically

$$(x_1y_1 + x_2y_2)^2 \leq (x_1^{\,2} + x_2^{\,2})(y_1^{\,2} + y_2^{\,2}) .$$

NOTE: This is a case of Schwarz's inequality, another form of which is

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 \leq (x_1^{\,2} + y_2^{\,2} + x_3^{\,2})(y_1^{\,2} + y_2^{\,2} + y_3^{\,2}) .$$

(b) Write these in vector notation.

(c) What geometric interpretation can be made for the case in which the left and right members are equal.

3-10. Summary and Review Exercises.

The chapter just concluded dealt with vectors and their applications. After reviewing some basic ideas about directed line segments (objects with both direction and magnitude), a vector was defined as an infinite set of equivalent directed line segments. The Origin-Principle allowed us to relate a vector to any point in space as an origin. We found it useful to select the origin-vector, that member of each set with its initial point at the origin, as the simplest representative of a vector. The unit vector and zero vector were defined and the term scalar introduced.

The next step in setting up an algebra of vectors was taken when the equality of vectors was defined in accordance with common practice. The operations of addition and subtraction of vectors and the product of a vector by a scalar were defined. The last concept made it possible to state that two vectors are parallel if and only if one is a scalar multiple of the other. The Origin-Principle related operations with vectors to the corresponding operations with their respective origin-vectors.

It was then proved that the commutative and associative laws hold for the addition of vectors. Scalar multiplication satisfied the associative law $(rs)\vec{P} = r(s\vec{P})$ and the distributive laws $r(\vec{P} + \vec{Q}) = r\vec{P} + r\vec{Q}$ and $(r + s)\vec{P} = r\vec{P} + s\vec{P}$. The zero vector $\vec{0}$ has the usual properties of the additive identity; the additive inverse, $-\vec{P}$, is defined by $\vec{P} + (-\vec{P}) = \vec{0}$.

The definition of a linear combination of vectors made it possible to prove some basic theorems about vectors. Theorem 3-5 stated that in a plane any vector can be expressed in terms of any two non-parallel and non-zero vectors. After the study of vector components, it was pointed out that any vector can be represented as a linear combination of the unit vectors

$i = [1,0]$ and $j = [0,1]$. Theorem 3-7 made it possible to determine if a point P lies on the line passing through the terminal points of two distinct vectors $\hat{A}$ and $\hat{B}$ which do not lie on the same line by proving that $\hat{P} = (1 - r)\hat{A} + r\hat{B}$. Sets of points on a given line could now be given a vector characterization. Theorem 3-8 offered a second method for dividing a line segment in a given ratio.

Vector components play a basic role in the application of vectors. The operations on vectors were defined in terms of these components. If $\hat{X} = [a,b]$, $\hat{Y} = [c,d]$, then $\hat{X} + \hat{Y} = [a + c\ ,\ b + d]$ and $r\hat{X} = [ra,rb]$.

The inner product of two vectors was defined by $\hat{X} \cdot \hat{Y} = |\hat{X}||\hat{Y}| \cos \theta$ where $\angle\theta$ is the angle between the two vectors, with $0 \leq \theta \leq \pi$. It was then proved that if $\hat{X} = [x_1,x_2]$ and $\hat{Y} = [y_1,y_2]$, then $\hat{X} \cdot \hat{Y} = x_1y_1 + x_2y_2$. A physical application was presented in the concept of work in physics. An important theorem is that two vectors, $\hat{X}$ and $\hat{Y}$, are perpendicular if and only if $\hat{X} \cdot \hat{Y} = 0$. The inner product has the following properties:

(1) $\hat{X} \cdot (\hat{Y} + \hat{Z}) = \hat{X} \cdot \hat{Y} + \hat{X} \cdot \hat{Z}$.

(2) $(t\hat{X}) \cdot \hat{Y} = \hat{X} \cdot (t\hat{Y}) = t(\hat{X} \cdot \hat{Y})$ where t is a scalar.

(3) $\hat{X} \cdot (a\hat{Y} + b\hat{Z}) = a(\hat{X} \cdot \hat{Y}) + b(\hat{X} \cdot \hat{Y})$ where a and b are scalars.

The inner product has many applications in geometry. We showed how it could be used to determine an angle between vectors, to find the area of the triangle determined by two vectors with a common initial point, to prove that the diagonals of a rhombus are perpendicular, and to show that the altitudes of a triangle are concurrent. The chapter concluded with a discussion of the resolution of vectors. This concept has considerable application in physical problems.

In the following chapter which deals with methods of proof in analytic geometry, there will be more proofs applying vector methods to geometric problems. In Chapter 8 there will be a brief introduction to vectors in a three dimensional space.

Review Exercises

1. If $\overrightarrow{A} = [3,-5]$, $\overrightarrow{B} = [-1,6]$, $\overrightarrow{C} = [2,3]$, find $\overrightarrow{X}$ in component form such that

   (a) $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{C} + \overrightarrow{X}$
   (b) $2\overrightarrow{A} + 3\overrightarrow{B} = 4\overrightarrow{C} + 5\overrightarrow{X}$
   (c) $2(\overrightarrow{A} - \overrightarrow{B}) = 3(\overrightarrow{C} - \overrightarrow{X})$
   (d) $\overrightarrow{A} + 2\overrightarrow{X} = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{X}$
   (e) $3(\overrightarrow{X} + \overrightarrow{B}) = 2(\overrightarrow{X} - \overrightarrow{C})$
   (f) $\overrightarrow{X} + 2(\overrightarrow{X} + \overrightarrow{A}) + 3(\overrightarrow{X} + \overrightarrow{B}) = 0$

2. Prove Theorem 3-3.

3. Prove Theorem 3-4.

4. Let $\overrightarrow{A} = [2,3]$, $\overrightarrow{B} = [3,-2]$, $\overrightarrow{C} = [-1,3]$. Find in component form, the single vector equal to

   (a) $2\overrightarrow{A} + 3\overrightarrow{B} - \overrightarrow{C}$
   (b) $\overrightarrow{A} - 2\overrightarrow{B} + 3\overrightarrow{C}$
   (c) $2(\overrightarrow{A} + \overrightarrow{B}) - 3(\overrightarrow{B} - \overrightarrow{C})$
   (d) $5(\overrightarrow{A} - \overrightarrow{C}) + 3(\overrightarrow{C} - \overrightarrow{A})$
   (e) $3(\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}) + 2(\overrightarrow{A} - \overrightarrow{B} + \overrightarrow{C})$
   (f) $5(\overrightarrow{C} - \overrightarrow{A} + \overrightarrow{B}) - 3(\overrightarrow{B} + \overrightarrow{A} - \overrightarrow{C})$

5. Use the values of $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, as in Exercise 4, and find $\overrightarrow{X}$ in component form so that

   (a) $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{C} + \overrightarrow{X}$
   (b) $2\overrightarrow{A} + 3\overrightarrow{B} = 4\overrightarrow{C} + 5\overrightarrow{X}$
   (c) $2(\overrightarrow{A} - \overrightarrow{B}) = 3(\overrightarrow{C} - \overrightarrow{X})$
   (d) $\overrightarrow{A} + 2\overrightarrow{X} = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{X}$
   (e) $3(\overrightarrow{X} + \overrightarrow{B}) = 2(\overrightarrow{X} - \overrightarrow{C})$
   (f) $\overrightarrow{X} + 2(\overrightarrow{X} + \overrightarrow{A}) + 3(\overrightarrow{X} + \overrightarrow{B}) = 0$

6. Use the values of $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, as in Exercise 4, and find the numerical value of

   (a) $\overrightarrow{A} \cdot \overrightarrow{B}$
   (b) $2\overrightarrow{A} \cdot 3\overrightarrow{B}$
   (c) $3\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C})$
   (d) $2\overrightarrow{B} \cdot (3\overrightarrow{A} + 2\overrightarrow{C})$
   (e) $(\overrightarrow{A} + \overrightarrow{B}) \cdot (\overrightarrow{A} - \overrightarrow{B})$
   (f) $(2\overrightarrow{B} + 3\overrightarrow{C}) \cdot (2\overrightarrow{B} - 3\overrightarrow{C})$
   (g) $(3\overrightarrow{A} + 5\overrightarrow{B}) \cdot (3\overrightarrow{B} - 2\overrightarrow{C})$
   (h) $(\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}) \cdot (\overrightarrow{B} - \overrightarrow{A} + \overrightarrow{C})$
   (i) $(2\overrightarrow{A} - 3\overrightarrow{B} + 4\overrightarrow{C}) \cdot (5\overrightarrow{A} - 2\overrightarrow{C} + 4\overrightarrow{B})$
   (j) $\overrightarrow{A} \cdot \overrightarrow{A} + \overrightarrow{B} . \overrightarrow{B} + \overrightarrow{C} \cdot \overrightarrow{C}$

7. Use the values of A , B , C , as in Exercise 4, and find the numerical values of

   (a) $|\overrightarrow{A}| + |\overrightarrow{B}|$
   (b) $|2\overrightarrow{A}| + |3\overrightarrow{C}|$
   (c) $2|\overrightarrow{A}| + 3|\overrightarrow{C}|$
   (d) $|3\overrightarrow{B}| - |4\overrightarrow{A}|$
   (e) $|\overrightarrow{A} - \overrightarrow{B}|$
   (f) $|2\overrightarrow{A} + 3\overrightarrow{C}|$
   (g) $|3\overrightarrow{B} - 4\overrightarrow{A}|$
   (h) $|\overrightarrow{A}|^2 - |\overrightarrow{B}|^2$
   (i) $|\overrightarrow{A}|^2 + |\overrightarrow{B}|^2 + |\overrightarrow{C}|^2$
   (j) $|2\overrightarrow{A}|^2 + |3\overrightarrow{B}|^2 + |4\overrightarrow{C}|^2$
   (k) $|2\overrightarrow{A} + 3\overrightarrow{B} + 4\overrightarrow{C}|^2$
   (l) $|\overrightarrow{A} - \overrightarrow{B}|^2$
   (m) $2|\overrightarrow{A}|^2 + 3|\overrightarrow{B}|^2 + 4|\overrightarrow{C}|^2$
   (n) $|\overrightarrow{A}|^2 + 2|\overrightarrow{A}|\,|\overrightarrow{B}| + |\overrightarrow{B}|^2$

8. If $i = [1,0]$ and $j = [0,1]$, we may express the vectors of Exercise 4 thus: $\vec{A} = 2i + 3j$, $\vec{B} = 3i - 2j$, $\vec{C} = -i + 3j$. In each part of Exercise 4, restate the original problem in terms of i and j; then, carry out your computations and express your results in terms of these components.

9. (Refer to Exercises 8 and 4 above.) Restate, in each part of Exercise 5, the problem and the solution in terms of i and j components.

10. (Refer to Exercises 8 and 4 above.) Restate, in each part of Exercise 6, the problem and the solution in terms of i and j components.

11. Given $A = (4,1)$, $B = (2,5)$, $C = (-2,3)$, and $D = (0,-4)$.

    (a) Find the angle measure of $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$; check your results.

    (b) Using O as the origin, find the areas of $\triangle OAB$, $\triangle OBC$, and $\triangle OAC$.

    (c) Use the results from part (b) to find the area of $\triangle ABC$.

12. Try to develop, with the methods of this chapter, a formula for the area of $\triangle ABC$, where $A = (a_1,a_2)$, $B = (b_1,b_2)$, $C = (c_1,c_2)$.

13. Find the area of the parallelogram in which $\overline{OA}$ and $\overline{OB}$ are adjacent sides. Can you apply these results to an earlier exercise in this set?

14. Find the vector representation of an exterior point of division which divides the directed segment (R,S) in the ratio $\frac{a}{b}$ where:

    (a) $\vec{R} = [2,-1]$, $\vec{S} = [-1,3]$, and $\frac{a}{b} = -2$

    (b) $\vec{R} = [-1]$, $\vec{S} = [2]$, and $\frac{a}{b} = -\frac{1}{2}$

    (c) $\vec{R} = [2,3,1]$, $\vec{S} = [1,-2,4]$, and $\frac{a}{b} = -3$

    (d) $\vec{R} = [-9,7]$, $\vec{S} = [3,-2]$, and $\frac{a}{b} = -\frac{1}{3}$

*15. Given the triangle ABC with $\vec{A} = [2,3]$, $\vec{B} = [-1,2]$, and $\vec{C} = [1,4]$.

    (a) Describe the triangular region, its interior, and the triangle itself, using these vectors and two scalars.

    (b) Show that $[1,3]$ is a vector whose terminal point is an interior point of the triangle.

    (c) Show that $[1,1]$ is a vector whose terminal point is an exterior point of the triangle.

    •(d) Show that the segment joining the points described in (b) and (c) intersects the triangle.

*16. Consider the convex quadrilateral ABCD with $\vec{A} = [2,3]$, $\vec{B} = [-1,2]$, $\vec{C} = [1,4]$, and $\vec{D} = [2,4]$. Find an expression for the polygonal region ABCD using these vectors and three scalars.

*17. Given the four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$, whose terminal points are not coplanar, find an expression for the tetrahedral region ABCD in terms of these vectors and three scalars.

18. Find the measure of the angles formed by the intersection of the lines

(a) $2x + 3y - 8 = 0$ and $3x - 2y + 4 = 0$.

(b) $5x + y - 2 = 0$ and $2x - y + 6 = 0$.

(c) $x + y + 3 = 0$ and

(d) $x + 2y = 0$ and $x = 4$

19. Points $A = (1,0)$, $B = (5,-2)$, and $C = (3,4)$ are the vertices of a triangle. Find the measure of each angle of $\triangle ABC$.

20. Given points $P = (-3,-8)$, $Q = (14,9)$, $R = (4,9)$, and $S = (-3,2)$. Find the measure of each angle of quadrilateral PQRS , and name the figure.

Chapter 4

PROOFS BY ANALYTIC METHODS

4-1. Introduction.

One of the satisfactions we hope you will gain from your study of analytic geometry is the realization that you have some very powerful tools for solving many seemingly difficult or impossible problems. We can demonstrate this, even so early in our work, by observing the simplicity and directness of analytic proofs for some theorems from plane geometry and trigonometry. You will recall many of these theorems, and you also may recall some of the struggles which resulted from using synthetic methods on these problems.

By increasing the number of methods available to solve problems, we create another problem--the uncertainty as to which method to use in a given situation. We shall sometimes ask you to use a particular method so that you may develop competence and confidence in its use. A tennis player may, in order to strengthen his backhand, be encouraged to use it temporarily more than he would in normal play. Your uncertainty and discomfort with a new method will last only until you have mastered it. You should understand also that even a competent mathematician may start with one method and discover later that it is not as convenient as another method. As you study the examples in this chapter, you should watch for clues to the reasons for choosing one method rather than another. Careful observation at this point will smooth the way as you proceed.

For the purposes of this chapter we assume that you know the kinds and basic properties of common geometric figures and that diagonals, medians, and the like, have been defined. These items, as well as the theorems to be discussed, may be reviewed in SMSG Geometry, Intermediate Mathematics, or some equivalent source.

4-2. Proofs Using Rectangular Coordinates.

Let us now prove some geometric theorems in rectangular coordinates.

Example 1. Prove: The median to the base of an isosceles triangle is perpendicular to the base. We might find the triangle placed in relation to the coordinate axes, as in Figure 4-1, with $\overline{AC} \cong \overline{BC}$ and with D the midpoint of $\overline{AB}$. From an analytic point of view, to prove $\overline{CD} \perp \overline{AB}$ we must show that the product of the slope of $\overline{AB}$ and the slope of $\overline{CD}$ is -1.

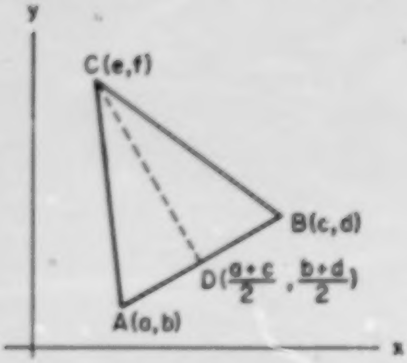


Figure 4-1

In order to ensure that the triangle is a general one we might select coordinates as follows: $A = (a,b)$, $B = (c,d)$, $C = (e,f)$. It follows that midpoint $D = (\frac{a+c}{2}, \frac{b+d}{2})$. By hypothesis $d(A,C) = d(B,C)$. We apply the distance formula to obtain

$$\sqrt{(a-e)^2 + (b-f)^2} = \sqrt{(c-e)^2 + (d-f)^2}\ ,$$

$$a^2 - 2ae + e^2 + b^2 - 2bf + f^2 = c^2 - 2ce + e^2 + d^2 - 2df + f^2\ , \text{ or}$$

$$a^2 - 2ae + b^2 - 2bf = c^2 - 2ce + d^2 - 2df\ . \tag{1}$$

We next calculate slopes. The slope of $\overline{CD}$ is $\dfrac{\frac{b+d-2f}{2}}{\frac{a+c-2e}{2}}$ and the slope of $\overline{AB}$ is $\frac{b-d}{a-c}$.

The product of the two slopes is

$$\frac{b^2 + bd - 2bf - bd - d^2 + 2df}{a^2 + ac - 2ae - ac - c^2 + 2ce} = \frac{b^2 - 2bf - d^2 + 2df}{a^2 - 2ae - c^2 + 2ce}\ .$$

Equation (1) can be written as

$$(2)\qquad a^2 - 2ae - c^2 + 2ce = -b^2 + 2bf + d^2 - 2df .$$

Substituting the right member of (2) into the denominator of the product of the slopes, we obtain

$$\frac{b^2 - 2bf - d^2 + 2df}{-b^2 + 2bf + d^2 - 2df} = -1 ;$$

hence, the theorem is proved.

It would be discouraging indeed if all of our coordinate proofs involved as much algebraic manipulation as exhibited in this example. Fortunately, this is not the case, and you may already see what can be done to simplify the algebra. It was not necessary to choose the coordinates as we did.

The properties of geometric figures depend upon the relations of the parts and not upon the position of the figure as a whole. Therefore, in our example, since only the triangle and not its location is specified, we could just as well select a coordinate system in which A is the origin and B lies on the positive side of the x-axis. This situation is illustrated in Figure 4-2. We now may have the following coordinates for the points: $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, $D = (\frac{a}{2},0)$. *Note that several of the coordinates are zero.* This is the feature which simplifies the algebra in our theorems, and this desirable goal provides us with a general guide in choosing coordinate axes for all our problems.

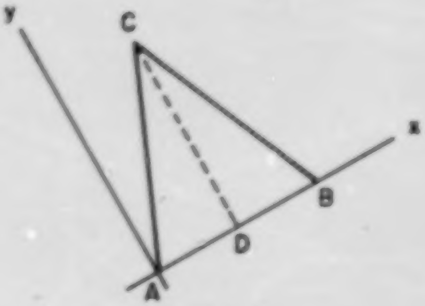


Figure 4-2

In actual practice we are more likely to make a drawing with the axes oriented as in Figure 4-3. This leads us to consider two methods of relating a geometric figure to a set of axes.

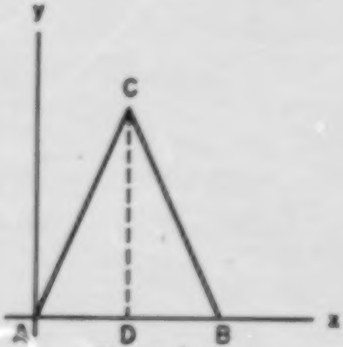


Figure 4-3

The method we have just described, that of assigning coordinates to a given geometric figure, is based upon the properties of coordinate systems developed in Chapter 2. Another method in common use employs the principles of rigid motion in which geometric objects are "moved" to more suitable locations without changing their size or shape. With respect to our current example, we would arrive at Figure 4-3 through this second method by assuming a fixed coordinate system upon which we place $\triangle ABC$ so that A coincides with the origin and B is placed on the positive side of the x-axis. The difference in the methods is largely one of viewpoint.

Another device which you will find useful can be illustrated by assigning coordinates to the vertices of $\triangle ABC$ in Figure 4-3 as follows: $A = (0,0)$, $B = (2a,0)$, $C = (b,c)$. The reason for using $2a$ for the abscissa of B is that we now have $D = (a,0)$, and we can complete the algebra without so much calculation involving fractions. The principle here is that a few minutes of foresight may save hours of patience.

Sometimes we pay a small price for the simplicity we gain. For example, the choice of coordinates suggested in the previous paragraph leads to trouble regarding the slopes. Although the slope of $\overline{AB}$ can be found to be zero, $\overline{CD}$ does not have a slope, since $a = b$. (Use the distance formula with $d(A,C) = d(B,C)$ to verify this.) Nevertheless, the problem has been simplified, for this means that $\overline{AB}$ is horizontal and $\overline{CD}$ is vertical, and this is also a condition for perpendicularity.

You might have chosen a coordinate system in which $\overline{AB}$ is on the x-axis but D is the origin. This is a fine choice. As you can see in Figure 4-4, if we choose $A = (a,0)$, then $B = (-a,0)$. It remains for us to prove that C lies on the y-axis. Let $C = (b,c)$ and use the distance formula in $d(A,C) = d(B,C)$. You can show that $b = 0$; hence, C lies on the y-axis and $\overline{CD} \perp \overline{AB}$.

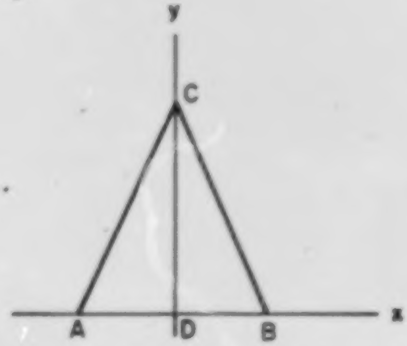


Figure 4-4

Let us summarize the procedures we have seen in this example. Usually there are more ways than one to attack any given problem, but certain general steps can be outlined. It was natural and useful in this example to use rectangular coordinates, since we were concerned with midpoints, lengths, and perpendicularity. Other situations we meet later may lead naturally to vectors or polar coordinates. In the cases for which we decide to use rectangular coordinates, we might follow the outline suggested below.

(a) Choose a coordinate system (or place the figure on one) so as to simplify the algebraic processes. Often this means having a vertex of the figure at the origin and one of its sides on the x-axis.

(b) Assign coordinates to points of the figure so as to accommodate the hypothesis as simply and clearly as possible. That is, make the figure sufficiently, but not unnecessarily, general.

(c) If possible, state the hypothesis and conclusion in a way that will correspond closely to the algebraic procedures being used.

(d) Plan an algebraic proof. Watch for opportunities to employ the distance, midpoint, and slope formulas.

Let us try another theorem from plane geometry.

Example 2. Prove: The diagonals of a parallelogram bisect each other.

Following the outline of our procedures, (a) to (c), we represent a parallelogram in a drawing and orient it with respect to the axes as in Figure 4-5. We let $A = (0,0)$ and $B = (a,0)$. The question of choosing coordinates for C and D can stand some discussion. The coordinates of C and D are not independent of those of A and B, nor are they independent of each other. How much can we assume about a parallelogram? We know by definition that the opposite sides of a parallelogram are parallel. This enables us to see at once that C and D have the same ordinate. Furthermore, since $\overline{BC} \parallel \overline{AD}$, their slopes are equal. This suggests that we use the slope formula to obtain a relation between the abscissas of C and D; namely, that the abscissa of C is the abscissa of B plus the abscissa of D. Thus we write $D = (b,c)$ and $C = (a + b,c)$. If we are allowed to use

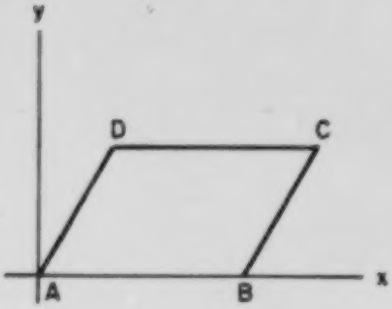


Figure 4-5

the property of a parallelogram that the opposite sides have equal lengths, then we shall reach the same conclusion more readily.

Some people prefer to employ these elementary properties of the common figures; others choose to assume no more than the definitions. For the purposes of this section we shall agree that we may use the properties ascribed to geometric figures by their definitions and by the theorems listed in Exercises 4-2, taking these theorems in the order in which they are listed. Our current example would be listed after Exercise 4 so the conclusion of Exercise 4 would be available to us when we chose coordinates for Figure 4-5.

The conclusion of our example is reached quickly. We are required to prove that the diagonals bisect each other. This means that each diagonal intersects the other at its midpoint. An application of the midpoint formula shows that the midpoint of each diagonal is $(\frac{a + b}{2}, \frac{c}{2})$.

We conclude this section with a challenge. Try to prove the following theorem by synthetic methods, and compare your proof with the one suggested below.

Example 3. Prove: If two medians of a triangle are congruent, the triangle is isosceles.

We prefer to use coordinates. The triangle must not be assumed to be isosceles, so we assign coordinates in Figure 4-6 as follows: $A = (2a,0)$, $B = (2b,0)$, $C = (0,2c)$. Let $M = (a,c)$ be the midpoint of $\overline{AC}$, and let $N = (b,c)$ be the midpoint of $\overline{BC}$. Next we shall express the hypothesis, $d(A,N) = d(B,M)$, in terms of the distance formula. You are encouraged to state the desired conclusion and to complete the details of the proof.

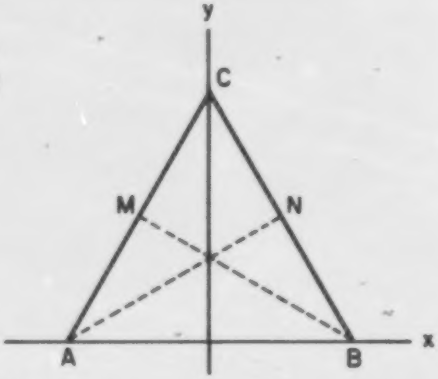


Figure 4-6

Exercises 4-2

The following exercises are theorems selected from the usual development of plane geometry. You are to prove these theorems in rectangular coordinates, using the "ground rules" we have outlined.

1. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to one-half the length of the third side.
2. If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.
3. The locus of points equidistant from two points is the perpendicular bisector of the line segment joining the two given points.
4. The opposite sides of a parallelogram have equal length.
5. If two sides of a quadrilateral have equal length and are parallel, the quadrilateral is a parallelogram.
6. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
7. If the diagonals of a parallelogram have equal length, the parallelogram is a rectangle.
8. The diagonals of a rhombus are perpendicular.
9. If the diagonals of a parallelogram are perpendicular, the parallelogram is a rhombus.
10. The line segments joining in order the midpoints of the successive sides of a quadrilateral form a parallelogram.
11. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.
12. The diagonals of an isosceles trapezoid have equal length.
13. The median of a trapezoid is parallel to the bases and has length equal to one-half the sum of the lengths of the bases.
14. If a line bisects one of the nonparallel sides of a trapezoid and is parallel to the bases, it bisects the other nonparallel side.

15. In any triangle, the square of the length of a side opposite an acute angle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of the length of one of the two sides and the length of the projection of the other on it.

16. The medians of a triangle are concurrent in a point that divides each of the medians in the ratio 2:1 .

17. The altitudes of a triangle are concurrent.

*18. A line through a fixed point P intersects a fixed circle in points A and B . Find the locus of the midpoint of $\overline{AB}$. (Consider three possible positions for P relative to the fixed circle.)

4-3. Proofs Using Vectors.

We shall now prove several theorems of geometry by vector methods. Some of the proofs are more difficult than those using methods discussed in your geometry course or in the preceding section. Others are accomplished more simply or concisely. In any case, the experience will be of great help in future mathematics courses and in applications to science or engineering. It will contribute toward your general ability to solve problems by giving you an additional tool and approach.

We shall demonstrate these approaches by solving several problems in detail.

Example 1. Prove that the median of a trapezoid is parallel to the bases and has length equal to one-half the sum of the lengths of the bases.

We first draw and label a trapezoid ABCD with $\overline{AB} \parallel \overline{CD}$ and with E and F the respective midpoints of $\overline{AD}$ and $\overline{BC}$. If we were using a rectangular coordinate system in this proof, we probably would choose the axes as in Figure 4-7. But since we are using a vector proof, we do not need the axes at all. In fact, because the origin vectors would not give us any advantage in the proof, neither do we specify an origin.

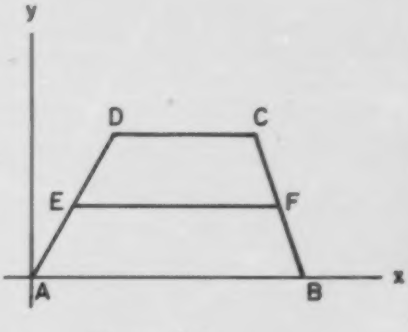


Figure 4-7

A vector drawing for the problem might then appear as in Figure 4-8.

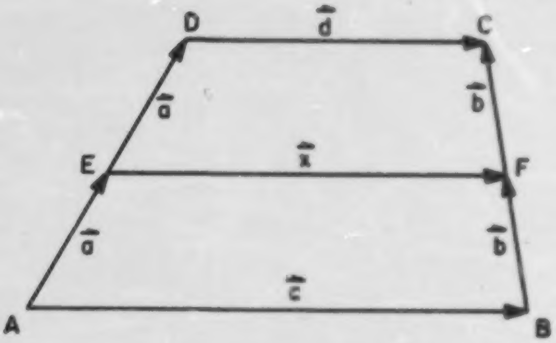


Figure 4-8

Something should be said about our choice of vector representation. Since E is the midpoint of $\overline{AD}$, if we represent $\overline{AE}$ by $\vec{a}$, then $\overline{ED}$ may also be represented by $\vec{a}$. Similarly, we choose $\vec{b}$ on the other non-parallel side. $\vec{c}$ and $\vec{d}$ represent the bases, and $\vec{x}$ represents median $\overline{EF}$. We are to prove

$$d(E,F) = \frac{1}{2}\Big(d(A,B) + d(C,D)\Big) \quad \text{and} \quad \vec{x} \parallel \vec{c} \quad \text{and} \quad \vec{x} \parallel \vec{d}\,.$$

Since one may "move" from E to F by going directly there, or by going through D and C, or by going through A and B, we have

$$\vec{x} = \vec{a} + \vec{d} - \vec{b}$$

and

$$\vec{x} = -\vec{a} + \vec{c} + \vec{b}\,;$$

therefore,

$$2\vec{x} = \vec{c} + \vec{d}\,.$$

Note again that when "moving" around a vector diagram, we add vectors which have the same sense of direction as our motion, and we subtract vectors which have the opposite sense of direction of our motion.

By the definition of parallel vectors, if $2\vec{x} = \vec{c} + \vec{d}$, then $\vec{x} \parallel (\vec{c} + \vec{d})$; since it is given that $\vec{c} \parallel \vec{d}$, it follows that $\vec{x} \parallel \vec{c}$ and $\vec{x} \parallel \vec{d}$. Furthermore, if $2\vec{x} = \vec{c} + \vec{d}$, then

$$|\vec{x}| = \frac{1}{2}\Big(|\vec{c}| + |\vec{d}|\Big) \quad \text{or} \quad d(E,F) = \frac{1}{2}\Big(d(A,B) + d(C,D)\Big)\;;$$

hence, the theorem is proved. You may wish to investigate what happens to the proof if you alter the direction of any of the vectors in the diagram.

Example 2. Show that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.

This situation is depicted by Figure 4-9 in which P, Q, R, and S are the given midpoints of the sides of quadrilateral ABCD. Once we choose an origin, each point of the figure determines an origin-vector. (It might be profitable for you to copy the figure on a piece of paper, select some point as an origin, and draw the origin-vectors to the vertices.)

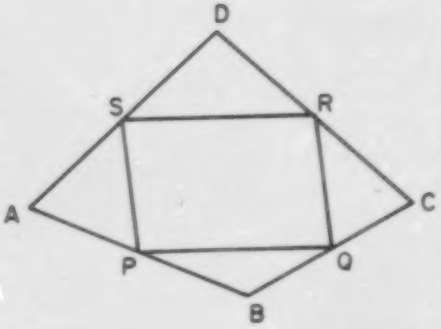


Figure 4-9

A portion of the figure with a set of origin-vectors is shown in Figure 4-10. We have also identified the vectors from A to P and from P to B in order to make use of the fact that $d(A,P) = d(P,B)$.

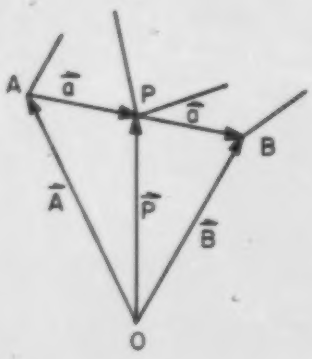


Figure 4-10

Since $\overrightarrow{P} = \overrightarrow{A} + \overrightarrow{a}$

and $\overrightarrow{P} = \overrightarrow{B} - \overrightarrow{a}$,

$$2\overrightarrow{P} = \overrightarrow{A} + \overrightarrow{B}$$

or $\overrightarrow{P} = \frac{1}{2}(\overrightarrow{A} + \overrightarrow{B})$.

Similarly, $\overrightarrow{Q} = \frac{1}{2}(\overrightarrow{B} + \overrightarrow{C})$,

$$\overrightarrow{R} = \frac{1}{2}(\overrightarrow{C} + \overrightarrow{D}),$$

$$\overrightarrow{S} = \frac{1}{2}(\overrightarrow{A} + \overrightarrow{D}).$$

(Had we not been interested in calling your attention to an application of vector addition, we would have obtained the same results from the Point of Division Theorem.)

We next note that vector $\vec{P} - \vec{Q}$ is equal to vector $\vec{S} - \vec{R}$ because both are equal to $\frac{1}{2}(\vec{A} - \vec{C})$. But why did we choose an expression like $\vec{P} - \vec{Q}$? There is a good reason for the choice. The line on vector $\vec{P} - \vec{Q}$ is parallel to $\overleftrightarrow{PQ}$, and remember that we are to show that certain segments are parallel.

In order to see the importance of $\vec{P} - \vec{Q} = \vec{S} - \vec{R}$, let us take a closer look at this situation, using a different origin. Suppose we isolate the lower part of Figure 4-9 containing points P, B, and Q as in Figure 4-11. If we choose B as the origin and E so that B is the midpoint of $\overline{QE}$, then we have vectors as marked on the diagram. The vector from Q to P is $-\vec{q} + \vec{p}$ which equals $\vec{P} - \vec{Q}$ and is therefore equal to $\vec{T}$. It follows then that the line on vector $\vec{P} - \vec{Q}$ is parallel to $\overleftrightarrow{PQ}$. Similarly the line on vector $\vec{S} - \vec{R}$ is parallel to $\overleftrightarrow{SR}$; and, since $\vec{P} - \vec{Q}$ is equal to and, consequently, parallel to $\vec{S} - \vec{R}$, we conclude that $\overline{PQ} \parallel \overline{SR}$. In the same way we show that $\overline{PS} \parallel \overline{QR}$, and PQRS is a parallelogram.

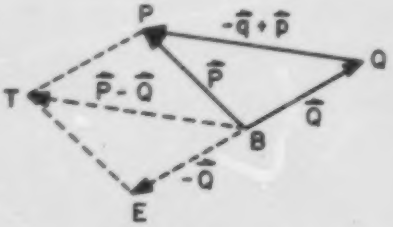


Figure 4-11

Example 3. Prove that the medians of a triangle intersect in a point which is a point of trisection of each median.

Solution. Let ABC be the triangle and P, Q, and R the midpoints of its sides as shown in Figure 4-12.

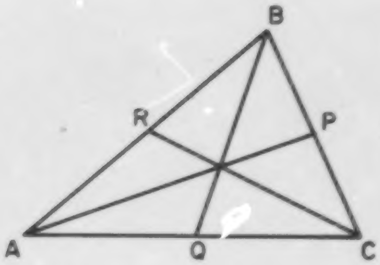


Figure 4-12

By the Origin Principle we may place the origin wherever we wish. If we are successful in proving the medians concurrent, the point of intersection would be an ideal choice for the origin, for then each origin-vector to a vertex would be collinear with the origin-vector to the midpoint of the opposite side.

We cannot assume all three medians concurrent, but we can let the origin O be the intersection of $\overline{AP}$ and $\overline{BQ}$. Then to prove that $\overline{CR}$ contains this point, we must prove that $\overset{\frown}{R}$ and $\overset{\frown}{C}$ are collinear, or that $\overset{\frown}{R}$ is a scalar multiple of $\overset{\frown}{C}$.

Proof. Let the origin be the intersection of $\overline{AP}$ and $\overline{BQ}$. Since P and Q are midpoints, and since $\overset{\frown}{P}$ and $\overset{\frown}{Q}$ are collinear with $\overset{\frown}{A}$ and $\overset{\frown}{B}$ respectively, we may write

(1) $\overset{\frown}{P} = \frac{1}{2}(\overset{\frown}{B} + \overset{\frown}{C}) = x\overset{\frown}{A}$

(2) $\overset{\frown}{Q} = \frac{1}{2}(\overset{\frown}{A} + \overset{\frown}{C}) = y\overset{\frown}{B}$.

If we subtract Equation (2) from Equation (1), we obtain

$$\overset{\frown}{P} - \overset{\frown}{Q} = \frac{1}{2}\overset{\frown}{B} - \frac{1}{2}\overset{\frown}{A} = x\overset{\frown}{A} - y\overset{\frown}{B} .$$

By the unique linear combination theorem (Theorem 3-5), $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$. The geometric interpretation of this discovery is that O is a trisection point of $\overline{AP}$ and $\overline{BQ}$. If we substitute these values in Equations (1) and (2) and add, we obtain

$$\overset{\frown}{P} + \overset{\frown}{Q} = \frac{1}{2}\overset{\frown}{A} + \frac{1}{2}\overset{\frown}{B} + \overset{\frown}{C} = -\frac{1}{2}\overset{\frown}{A} - \frac{1}{2}\overset{\frown}{B} .$$

Since $\overset{\frown}{R} = \frac{1}{2}(\overset{\frown}{A} + \overset{\frown}{B})$, the second two members of this equality become

$$\overset{\frown}{R} + \overset{\frown}{C} = -\overset{\frown}{R} \quad \text{or} \quad \overset{\frown}{R} = -\frac{1}{2}\overset{\frown}{C} .$$

Thus, R and C are collinear, O is on $\overline{CR}$, and O is a point of trisection of $\overline{CR}$.

If we choose another point as origin and let G be the point of intersection of the medians, the Point of Division Theorem permits us to write

$$\overset{\frown}{G} = \frac{1}{3}\overset{\frown}{A} + \frac{2}{3}\overset{\frown}{P} ,$$

or $$\overset{\frown}{G} = \frac{1}{3}\overset{\frown}{A} + \frac{2}{3}(\frac{1}{2}\overset{\frown}{B} + \frac{1}{2}\overset{\frown}{C}) = \frac{1}{3}\overset{\frown}{A} + \frac{1}{3}\overset{\frown}{B} + \frac{1}{3}\overset{\frown}{C} = \frac{1}{3}(\overset{\frown}{A} + \overset{\frown}{B} + \overset{\frown}{C}) .$$

We have not only solved the problem, but also have represented the point of concurrency by the vector $\frac{1}{3}(\overset{\frown}{A} + \overset{\frown}{B} + \overset{\frown}{C})$. This point is called the centroid of the triangle and has an important property connected with the idea of the center of gravity of a physical object. If a thin uniform sheet (such as cardboard) is cut in the shape of the triangle, it can be balanced on a pencil point placed at the point corresponding to the centroid.

Example 4. Show that the bisector of an angle of a triangle divides the opposite side into segments whose lengths are proportional to the lengths of the adjacent sides.

Solution. Let $\overline{PT}$ bisect $\angle QPR$, and let the vector from P to Q be represented by $\overset{\frown}{a}$, the vector from P to T by $\overset{\frown}{b}$, and the vector from P to R by $\overset{\frown}{c}$, as shown in Figure 4-13. We are to show that

$$\frac{d(R,T)}{d(T,Q)} = \frac{d(P,R)}{d(P,Q)} .$$

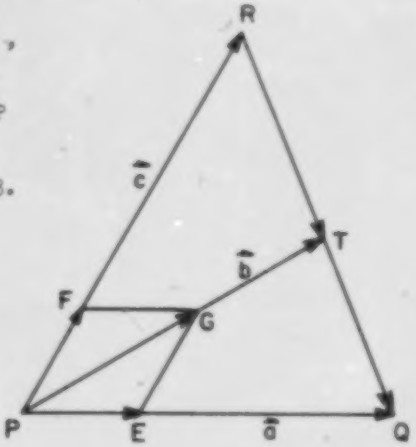


Figure 4-13

This problem involving an angle bisector affords us an opportunity to demonstrate the use of unit vectors in a solution. A vector which bisects the angle between $\overset{\frown}{a}$ and $\overset{\frown}{c}$ must lie along the diagonal of a rhombus whose adjacent sides lie along $\overset{\frown}{a}$ and $\overset{\frown}{c}$. We employ unit vectors to accomplish this result.

Any vector along $\overset{\frown}{a}$ can be represented as a scalar multiple of $\overset{\frown}{a}$. In particular, the unit vector along $\overset{\frown}{a}$ can be represented by $\frac{1}{|\overset{\frown}{a}|}\overset{\frown}{a}$ or $\frac{\overset{\frown}{a}}{|\overset{\frown}{a}|}$. Then the vector from P to E , $\frac{\overset{\frown}{a}}{|\overset{\frown}{a}|}$, and the vector from P to F , $\frac{\overset{\frown}{c}}{|\overset{\frown}{c}|}$, determine a rhombus whose diagonal $\overline{PG}$ bisects the angle determined by $\vec{a}$ and $\vec{c}$. The vector from P to G is then $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}$, and any vector along it, say from P to T, can be represented by a scalar multiple

$k\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$.

Now suppose r is the ratio $\frac{d(R,T)}{d(R,Q)}$. Since the vector from R to Q is $(\vec{a} - \vec{c})$, the vector from R to T may be expressed as $r(\vec{a} - \vec{c})$, and that from T to Q by $(1 - r)(\vec{a} - \vec{c})$. We may write

$$\vec{b} = \vec{c} + r(\vec{a} - \vec{c})$$

and obtain $k\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right) = \vec{c} + r(\vec{a} - \vec{c})$, or $\frac{k}{|\vec{a}|}\vec{a} + \frac{k}{|\vec{c}|}\vec{c} = r\vec{a} + (1 - r)$

Equating the corresponding coefficients, we have

$$\frac{k}{|\vec{a}|} = r \quad \text{and} \quad \frac{k}{|\vec{c}|} = 1 - r .$$

It follows that

$$\frac{r}{1 - r} = \frac{|\vec{c}|}{|\vec{a}|} ;$$

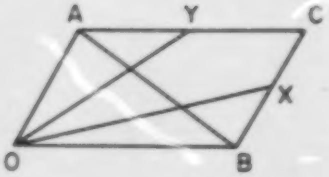
hence,

$$\frac{d(R,T)}{d(T,Q)} = \frac{d(P,R)}{d(P,Q)} .$$

Exercises 4-3

1. Give a vector proof that the diagonals of a parallelogram bisect each other.

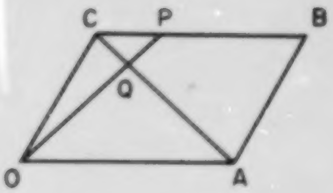
2. Prove by using vectors that a line segment which joins one vertex of a parallelogram to the midpoint of an opposite side passes through a point of trisection of a diagonal. ($\overline{AB}$ in the figure.) Prove also that the diagonal $\overline{AB}$ passes through points of trisection of $\overline{OX}$ and $\overline{OY}$.



3. Rework Example 3 for the case in which the origin is selected to be the point A. Does this choice of origin simplify the proof?

4. In parallelogram OABC , $\overline{OP}$ intersects $\overline{AC}$ at Q .

If $\frac{d(C,P)}{d(C,B)} = \frac{1}{r}$, show that $\frac{d(C,Q)}{d(C,A)} = \frac{1}{r+1}$.



Exercises 5 to 10 are theorems from plane geometry which you are to prove by the vector methods illustrated in the examples of this section.

5. If two medians of a triangle have equal length, then the triangle is isosceles.

6. The median to the base of an isosceles triangle is perpendicular to the base.

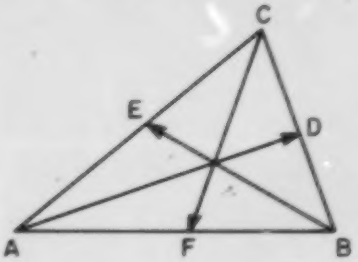
7. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

8. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to one-half the length of the third side.

9. An angle inscribed in a semicircle is a right angle.

10. The bisectors of a pair of adjacent supplementary angles form a right angle.

11. D, E, and F are midpoints of $\triangle ABC$, as shown. Let the vector from A to D be $\vec{a}$, the vector from B to E be $\vec{b}$, the vector from C to F be $\vec{c}$. Prove that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.



4-4. Proofs Using Polar Coordinates.

Polar coordinates are useful in many applications, particularly if the problems involve rotations or trigonometric functions.

The following example from trigonometry illustrates one such use.

Example 1. Show that $\cos(\beta - \alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha$.

Let $\angle\alpha$ and $\angle\beta$ be as shown in Figure 4-14. We select points B and C on the respective terminal sides of the angles and let $d(B,C) = a$, $d(A,C) = b$, and $d(A,B) = c$. The distance formula tells us that

$$(1)\quad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 .$$

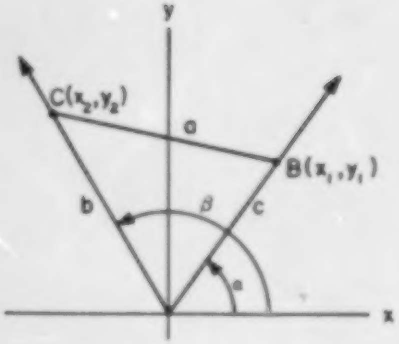


Figure 4-14

Now if we convert from rectangular to polar coordinates as outlined in Section 2-5, Equation (1) becomes

$$a^2 = (b\cos\beta - c\cos\alpha)^2 + (b\sin\beta - c\sin\alpha)^2 .$$

Expanding the right member and applying the identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$(2)\quad a^2 = b^2 + c^2 - 2bc(\cos\beta \cos\alpha + \sin\beta \sin\alpha) .$$

Noting that the measure of $\angle BAC = \beta - \alpha$ and comparing Equation (2) with the Law of Cosines for $\triangle ABC$, we see that

$$\cos(\beta - \alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha .$$

As for the next example, it is unlikely that anyone would choose this kind of proof when other proofs are available, but nevertheless, it may be instructive to look at one demonstration of a simple geometric proposition using polar coordinates.

Example 2. Prove that the median to the base of an isosceles triangle bisects the vertex angle.

Consider Figure 4-15, in which $\overline{AC} \cong \overline{BC}$. In order to describe the angles in question, we let C be the pole. We also let D, the midpoint of $\overline{AB}$, lie on the polar axis. Without loss of generality, we have $A = (r, \alpha)$, $B = (r, \beta)$. We must prove $\alpha = -\beta$.

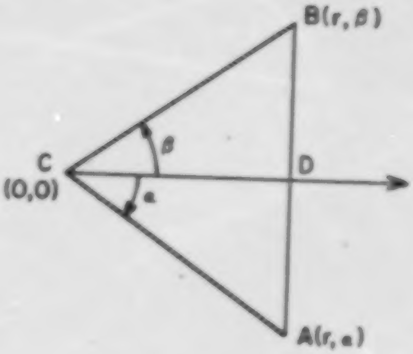


Figure 4-15

To simplify the notation we shall let $d(C,D) = f$ and $d(A,D) = d(B,D) = g$. Applying the Law of Cosines, we have,

in $\triangle BCD$, $g^2 = r^2 + f^2 - 2rf \cos \beta$,

and in $\triangle ACD$, $g^2 = r^2 + f^2 - 2rf \cos \alpha$.

We see then that $\cos \alpha = \cos \beta$. Since $0 < \alpha < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \beta < 0$, this implies $\alpha = -\beta$.

4-5. Choice of Method of Proof.

It is time we paused to survey the variety of problem-solving tools which are now at our disposal. We have a choice of three basic systems --rectangular coordinates, polar coordinates, and vectors; within each system we have different representations to suit different purposes. But the question uppermost in your mind at the moment probably is, "How do I decide which method is the best one to use?"

The question does not have a simple answer. Some problems are best worked by one particular method, other problems seem to be approachable by any of these methods, and some problems appear to be impossible regardless of what we try.

However, there are certain guidelines which may help us.

(1) Try to decide upon a coordinate system which is appropriate to the problem. Think over what is known about the problem, or what is to be proved, or what kind of answer is required.

   (a) Distances between points, slopes of lines, and midpoints of segments are easily handled in rectangular coordinates; therefore, when these ideas are present, you should try to fit rectangular coordinate axes to the problem.

   (b) If the problem involves angular motion or circular functions, it would be wise to look at the possibilities of polar forms.

   (c) Vectors are quite versatile and fit a wide range of conditions. Concurrence, parallelism, and perpendicularity of lines, as well as problems of physical forces, are situations which might lead you to choose a vector approach.

(2) Make a drawing relating the known facts of the problem to your choice of method. Much time and effort may be saved by a reasonably accurate drawing. This not only helps to relate the parts of the problem, but it serves as a check on the calculated results.

(3) Choose coordinates or vectors so as to simplify the algebra. Take advantage of all the given information at this stage, but be careful that you maintain generality where it is required.

(4) Watch for opportunities to use parametric representations. This may be something new to you, but you will observe frequent cases in succeeding chapters in which this special method will simplify troublesome problems.

(5) Work many, many problems. It also will help if you try to solve a given problem in several different ways. In this area of mathematics, experience is probably the most valuable asset. Sometimes a choice of method can be explained only on the basis of experience.

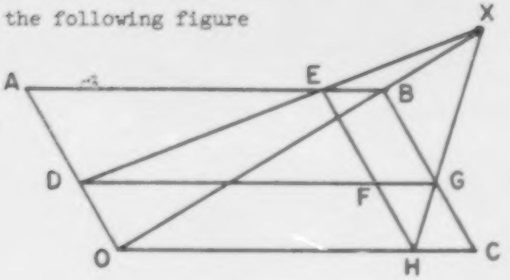
(6) After you have completed your solution to a problem, it is wise to look back over your work. You may see an unnecessary step you can eliminate, an unwarranted assumption you should justify, or a general tightening up you may accomplish. In any case, you gain a new perspective on your work which increases your understanding and appreciation of what you have done.

Review Exercises

For Exercises 1 to 10, first choose a coordinate system which you think is appropriate for each theorem, and then prove the theorem accordingly.

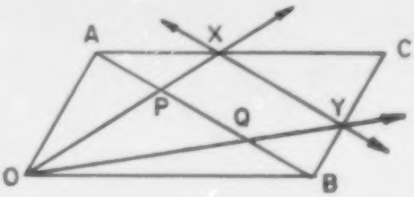
1. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.
2. The locus of the vertex of a right angle, the sides of which pass through two fixed points, is a circle.
3. The diagonals of a rectangle have equal length.
4. Show that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of its diagonals.
5. The line segments joining in order the midpoints of the successive sides of an isosceles trapezoid form a rhombus.
6. The line segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases and has length equal to one-half the difference of the lengths of the bases.
7. If lines are drawn through a pair of opposite vertices of a parallelogram and through the midpoints of a pair of opposite sides in such a way that the lines intersect one of the diagonals in distinct points, the lines are parallel and the diagonal is trisected.
8. The perpendicular bisectors of the sides of a triangle are concurrent in a point that is equidistant from the three vertices of the triangle.
9. If two sides of a triangle are divided in the same ratio, the line segment joining the points of division is parallel to the third side and is in the same ratio to it.
10. Show that the vector joining the midpoints of two opposite sides of a vector quadrilateral is equal to half the vector sum of the other two sides.

12. In the following figure



$OABC$, $DAEF$, and $HFGC$ are each parallelograms. Prove that the respective diagonals of the parallelograms $\overline{OB}$, $\overline{DE}$, and $\overline{HG}$, extended as necessary, meet in a single point X .

13. In parallelogram $OACB$, let P and Q be points on diagonal $\overline{AB}$ such that $d(A,P) = d(B,Q)$. Let $\overrightarrow{OP}$ intersect $\overleftrightarrow{AC}$ at X , and let $\overrightarrow{OQ}$ intersect $\overleftrightarrow{BC}$ at Y . Show that $\overleftrightarrow{XY} \parallel \overleftrightarrow{AB}$.

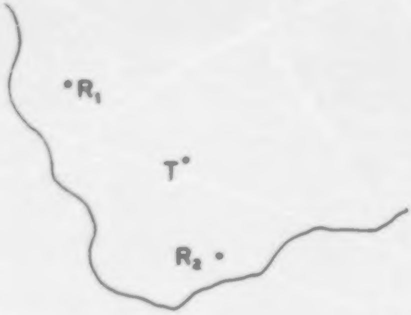


14. Prove that the sum of the squares of the lengths of the sides of a quadrilateral exceeds the sum of the squares of the lengths of its diagonals by 4 times the square of the length of the line segment that joins the midpoints of the diagonals.

15. A band of pirates buried their treasure on an island. They chose a spot at which to bury it in the following manner: Near the shore there were two large rocks and a large pine tree. One pirate started out from one rock along a line at right angles to the line between this rock and the tree. He marched a distance equal to the distance between this rock and the tree. Another pirate started out from the second rock along a line at right angles to the line between this second rock and the tree and marched a distance equal to the distance between this rock and the tree.

The rest of the band of pirates then found the spot midway between these two and there buried the treasure.

Many years later, these directions came to light and a party of treasure-seekers sailed off to find the treasure. When they reached the island, they found the two rocks with no difficulty. But the tree had long since disappeared, so they did not know how to proceed. All seemed lost till the cabin boy, who had just finished his freshman year at Yale, spoke up. Remembering the analytic geometry he had studied, he calculated where the treasure must be, and a short spell of digging proved him correct. How did he do it?



Chapter 5

GRAPHS AND THEIR EQUATIONS

5-1. Introduction

In Section 2-2 we discussed sets of points and their analytic representations. The relation between the two is at the heart of analytic geometry, and we shall review the fundamental notions briefly here. We confine the discussion to the plane, but the extension to space is immediate. The sets of points will frequently be the geometric figures we met earlier, and the analytic representations will usually be given in algebraic or trigonometric forms that we have met before. We propose to relate these ideas with the hope that your competence and appreciation for their use will continue to grow.

Let S be a set of points in a plane with a rectangular coordinate system. Let $s(x,y)$ be an open sentence involving two variables. Let S consist of those points (a,b) of the plane such that $s(a,b)$ is true. Then we say S is the *locus* (or *graph*) of the condition $s(x,y)$, and $s(x,y)$ is a *condition* for the set S. The plural of "locus" is "loci". (It is pronounced as though it were spelled "low-sigh". The rectangular coordinate system in the plane could be replaced by any other coordinate system appropriate to the problem and to the space in which we are working. The choice of a coordinate system determines the "language" in which the open sentence is stated. We shall often be concerned with the limitations of a particular language, and the details of the translation from one language to another.

Some of you may be used to a different way of talking about the matter. In the SMSG *Geometry* there is a discussion of characterizations of sets. A condition is said to characterize a set if every point in the set satisfies the condition and every point that satisfies the condition is in the set. The conditions we are chiefly interested in here are analytic conditions (conditions on the coordinates of points), whereas in *Geometry* the conditions were stated in geometric terms.

5-2. Conditions for Loci or Graphs, and Graphs of Conditions

The discussion above is quite general, but in practice the conditions that matter most are equations and inequalities. For example, we define the graph of an equation (inequality) in x and y to be the set of points whose coordinates satisfy the equation (inequality). Thus the locus of the equation $x^2 + y^2 = 4$ is the circle with center $(0,0)$ and radius 2, while the locus of the inequality $xy < 0$ is the set of points in the second quadrant or in the fourth quadrant. Using set notation these two loci can be expressed as follows:

$$\{P = (x,y) : x^2 + y^2 = 4\} ,$$

$$\{P = (x,y) : xy < 0\} .$$

Using the same notation we can express the loci of the equation $f(x,y) = 0$, and the inequality $g(x,y) > 0$ as follows:

$$\{P = (x,y) : f(x,y) = 0\} ,$$

$$\{P = (x,y) : g(x,y) > 0\} .$$

We now take up the problem of finding an analytic condition for a set of points in a plane. There is no routine procedure for doing this, but the following advice may be useful.

First a word about the choice of coordinate systems. When the terms of the problem leave you free, think carefully about the coordinate system to use. Some curves with complicated equations in rectangular coordinates have nice parametric representations. An equation in rectangular coordinates for a certain curve may be simpler than it is otherwise if a coordinate axis is an axis of symmetry. A circle of radius 3 has a simple equation in rectangular coordinates if its center is made the origin, a still simpler equation in polar coordinates if its center is chosen as the pole.

Following common usage we will use x and y for rectangular coordinates, and r and θ for polar coordinates. We will also assume in each case, unless otherwise specified, suitable choices of axes and units. Only with these assumptions may we speak about "the" locus of an equation. Without such assumptions an equation may have several quite different graphs, depending on our choices of coordinate systems. These matters will be considered more fully later, particularly in Chapter 6.

After choosing a coordinate system we can attack the problem. We start with a given set of points. These points are not given to us in a basket but instead are determined by some geometric condition. We are looking for an equivalent condition in terms of the coordinates of points. Let us look at what we do in several examples.

Example 1. We describe certain sets of points of the plane. You are asked to give analytic description of each set.

(a) All the points of the x-axis.

Solution. $\{P = (x,y) : y = 0\}$.

(b) All the points above the x-axis.

Solution. $\{P = (x,y) : y > 0\}$.

(c) All the points of the plane except those on either axis.

Solution. $\{P = (x,y) : xy \neq 0\}$.

(d) The midpoints of all line segments in the first quadrant which, with the coordinate axes, form a triangle whose area has a measure of 12 square units.

Solution. If $P = (x,y)$ is one such point, the endpoints of its segment have coordinates $(2x,0)$ and $(0,2y)$. The triangular region will then have area $\frac{1}{2}(2x)(2y)$, which must equal 12 . We have the simpler equivalent relationship $xy = 6$. The graph of this relationship contains points in the first and third quadrants but we want only those with positive coordinates. Thus, our answer is $\{P = (x,y) : xy = 6 , x > 0 , y > 0\}$.

Example 2. Find an equation in rectangular coordinates of the locus of all points equidistant from two distinct points.

Solution. Let the x-axis be the line through the two points and let the origin be the midpoint of the segment determined by them. Then the two points are $(a,0)$ and $(-a,0)$. Let (x,y) be any point in the plane. Then the distances to (x,y) from $(a,0)$ and $(-a,0)$ are $\sqrt{(x - a)^2 + y^2}$ and $\sqrt{(x + a)^2 + y^2}$, respectively. The point (x,y) belongs to our locus if and only if these two distances are equal, that is, if and only if

$$\sqrt{(x + a)^2 + y^2} = \sqrt{(x - a)^2 + y^2} . \tag{1}$$

Thus (1) is an equation of the locus. (1) is, of course, not the simplest possible equation for the locus. What is, and how can you get it from (1) ?

Example 3. We present some analytic descriptions of sets of points of the plane. Describe these sets in ordinary English.

(a) $\{P = (r,\theta) : r > 5\}$.

Solution. All points outside a circle whose center is at the pole and whose radius is 5 .

(b) $\{P = (x,y) : |x - 3| = 7\}$.

Solution. All the points on two parallel lines. These lines are parallel to the line $x = 3$, and lie one on each side of it and 7 units away.

(c) $\{P = (x,y) : xy + 2x - y > 2\}$.

Solution. This inequality may be written $xy + 2x - y - 2 > 0$, or $(x - 1)(y + 2) > 0$. This statement will be true for values of x and y such that either:

$$x - 1 > 0 \text{ and } y + 2 > 0 \text{ , or } x - 1 < 0 \text{ and } y + 2 < 0 \text{ ;}$$

that is if either:

$$x > 1 \text{ and } y > -2 \text{ , or } x < 1 \text{ and } y > -2 \text{ .}$$

The points we want lie in two "quadrants", as indicated in Figure 5-1. The graph does not include the boundaries of the regions. How could you change the analytic descriptions of the set to include these boundaries?

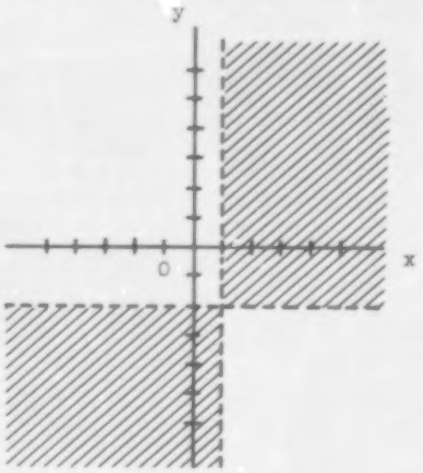


Figure 5-1

(d) $\{P(x,y) : |x + 1| < 3 \text{ and } |y + 1| \leq 4\}$.

Solution. All the points of a rectangular region, with center at the point $(-1,-1)$. The region is 6 units wide and does not include the vertical boundaries; it is 8 units high and does include the horizontal boundaries. It is pictured in Figure 5-2. We note that the corners of the region are not points of the graph.

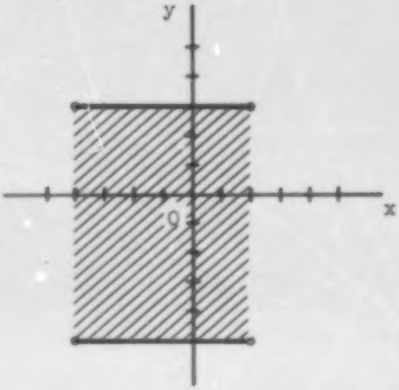


Figure 5-2

(e) $\{P = (r,\theta) : |r - 5.0| < .1\}$.

Solution. The set of points of the annular region between two concentric circles centered at the pole. The inner circle has radius 4.9 and the outer circle has radius 5.1 , but neither circle is part of the locus, which is illustrated in Figure 5-3.

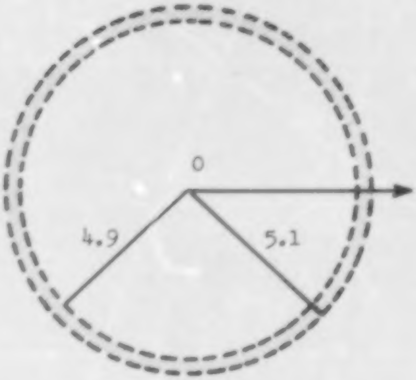


Figure 5-3

We have been using set notation because we wanted to be perfectly clear. Hereafter we shall be less formal. We might state the problem of Exercise 3(e): Describe and draw the graph of $|r - 5.0| < .1$.

Example 4. Find an equation in rectangular coordinates for the locus of all points which are equidistant from a given point F and a given line L .

Solution. The geometric condition for the locus defines a parabola, whose equation we now derive from the condition. With this in mind we let the line through F perpendicular to L be the y-axis, with the origin at the midpoint of the segment determined by F and the point where the perpendicular intersects L . (If F is in L , we pick F as the origin and leave the further details in this case as an exercise.) Finally, we let the y-coordinate of F be $\frac{p}{2}$, where $p \geq 0$. Then $F = (0,\frac{p}{2})$ and L is the line $y = -\frac{p}{2}$.

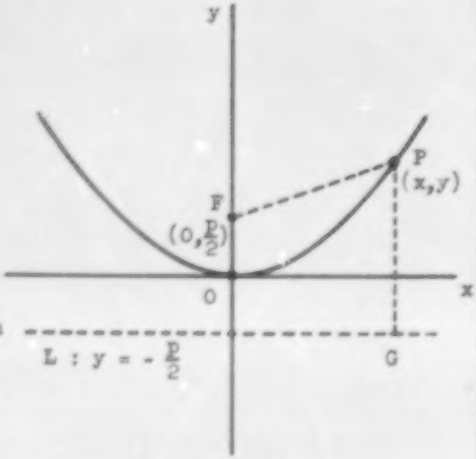


Figure 5-4

Let P = (x,y) be an arbitrary point in the plane. Then the things talked about in the geometric condition are the distances from P to F and to L . Using the distance formula we find that the first of these is $\sqrt{x^2 + (y - \frac{p}{2})^2}$. The second is $|y + \frac{p}{2}|$. The geometric condition says these two distances are to be equal. Hence

$$\sqrt{x^2 + (y - \frac{p}{2})^2} = |y + \frac{p}{2}| \tag{2}$$

is an equation for the locus. This is a complete solution of the original problem, but a simpler equation can be found. If we square both members of (2) and combine terms, we get the equation

$$x^2 = 2py . \tag{3}$$

There remains the question of whether (2) and (3) are equivalent. The only operation we have performed which might have caused trouble was the squaring of both sides. But any point on the locus of (2) is on the locus of the equation obtained by squaring both members of (2) , and hence on the locus of (3) . That the reverse is also true can be shown most simply by considering a more general problem. Let (a,b) be a point on the locus of $\left(f(x,y)\right)^2 = \left(g(x,y)\right)^2$, so that $\left(f(a,b)\right)^2 = \left(g(a,b)\right)^2$. Then $f(a,b) = \pm g(a,b)$. Now suppose, further, that if (x,y) is in the domains of f and g , then $f(x,y) \geq 0$ and $g(x,y) \geq 0$. We cannot have $f(a,b) = -g(a,b)$ unless both are zero, and hence $f(a,b) = g(a,b)$. Thus $\left(f(x,y)\right)^2 = \left(g(x,y)\right)^2$ and $f(x,y) = g(x,y)$ are equivalent equations. This result settles our question for us, since both members of (2) are non-negative for all x and y .

Example 5. A Coast Guard cutter, searching for a boat in distress, travels in a path with the property that the distance (in miles) of the cutter from its starting point, O , is equal to the radian measure of the angle generated by the ray from O to the cutter. Find an equation of the path in a suitable coordinate system. (Assume the surface of the ocean is a plane.)

Solution. The description of the path suggests that we should use polar coordinates, with O as pole and the polar axis in the direction in which the cutter is heading when it starts its search. If we do this we get immediately the function defined by the equation $r = \theta$. (By choosing the positive direction of rotation properly we can make θ positive.)

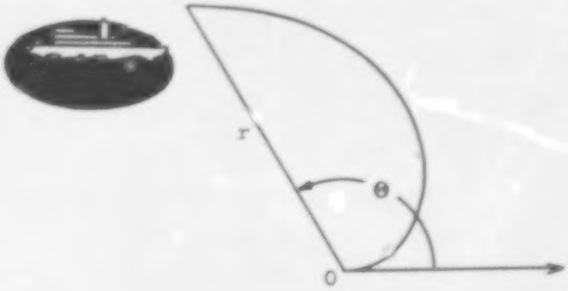


Figure 5-5

The path is a spiral.

If we use rectangular coordinates we get a much more complicated equation. Furthermore, no matter how we choose the axes, the equation does not define a function. Can you explain why not?

Related Polar Equations. In writing an analytic description of a set of points we may use to our advantage the freedom we have in choosing the type of coordinate system, the placement of the axes, and the units. In the case of polar coordinates there is an ambiguity imposed on us by the fact that each point now has infinitely many pairs of coordinates. This makes some matters easy, and some difficult. If a moving point traces and retraces its path in a recurrent pattern, a polar equation for the locus can represent this pattern, since (r,θ) and $(r,\theta + 2\pi n)$ are, for integral values of n , coordinates for the same point. On the other hand, since (r,θ) and $(-r,\theta + \pi)$ are also coordinates for the same point, we cannot avoid a certain ambiguity in writing equations of loci in polar coordinates. A point (r_1,θ_1) on the curve represented by the equation $r = f(\theta)$ also has the coordinates

$(-r_1, \theta_1 + \pi)$. If we substitute the latter coordinates in the equation we obtain the equation $-r_1 = f(\theta_1 + \pi)$ which may be written $r_1 = -f(\theta_1 + \pi)$. That is, every point of the curve represented by $r = f(\theta)$ is at the same time a point of the curve represented by $r = -f(\theta + \pi)$. We will call these equations,

$$\begin{cases} r = f(\theta) , \\ r = -f(\theta + \pi) , \end{cases}$$

related polar equations for the curve. In some cases these related polar equations are quite different in appearance and it takes some experience to recognize that they represent the same curve. On the other hand the related polar equations may be identical.

Example 6. The related equation for $r = 5 \sin \theta$ is $r = -5 \sin(\theta + \pi)$ $= -5(-\sin \theta) = 5 \sin \theta$, and is the same as the original equation.

Example 7. The related equation for $r = 3 \tan \theta$ is $r = -3 \tan(\theta + \pi) = -3 \tan \theta$, and is different from the original equation.

Example 8. The related equation for $r = 3(1 + \sin \theta)$ is $r = -3(1 + \sin(\theta + \pi)) = -3(1 - \sin \theta) = 3(\sin \theta - 1)$, and is different from the original equation.

Example 9. The related equation for $r = 5$ is $r = -5$, and is different from the original equation.

Because the correspondences between points and their polar coordinates and between sets of points and their representations in polar coordinates are not unique, we must define the graph of a polar equation to be not the set of points whose coordinates satisfy that equation but rather the set of points each of which has some pair of coordinates that satisfy the equation.

Exercises 5-2

For each of the following, write an equation or statement of inequality of the locus of a point which satisfies the stated condition. Use the coordinate system you think appropriate if one is not specified. If you use polar coordinates, give the pair of related equations in each case.

1. A point 3 units above the x-axis.
2. A point 5 units to the left of the y-axis.
3. A point equidistant from the x- and y-axes.
4. A point twice as far from the x-axis as it is from the y-axis.
5. A point a units from the origin.
6. A point a units from the point $(3,-2)$.
7. A point equidistant from $(3,0)$ and $(-5,0)$.
8. A point equidistant from $(2,3)$ and $(5,-4)$.
9. A point equidistant from the lines with equations $x + y - 2 = 0$ and $x + 2y + 2 = 0$.
10. A point whose distance from the line with equation $x + 2 = 0$ is equal to its distance from the point $(2,0)$.
11. A point whose distance from the line with equation $2x + y + 2 = 0$ is equal to its distance from the point $(2,-1)$.
12. A point the sum of whose distances from the points $(4,0)$ and $(-4,0)$ is 10.
13. A point the difference of whose distances from the points $(4,0)$ and $(-4,0)$ is 6.
14. A point the ratio of whose distances from the lines $2x + y - 4 = 0$ and $3x - y + 1 = 0$ is 2 to 3.
15. A point that is contained in the line through the points $(-1,2)$ and $(5,7)$.
16. A point, the product of whose distances from two fixed points is a constant. (This locus is called Cassini's Oval; it was studied by Giovanni Domenico Cassini in the late seventeenth century in connection with the motions of the earth and the sun.)
17. A point within 3 units distance from the x-axis.
18. A point at least 5 units distant from the origin.
19. A point no more than 1 unit from the y-axis.
20. A point no more than 2 units from $(1,3)$.
21. A point no nearer to the origin than it is to the point $(0,5)$.
22. A point no nearer to the origin than it is to the line $y = 4$.

23. A point nearer to the origin than to any point on the line $x = 10$.

24. A point between the lines $x = 6$, $x = -6$.

25. A point within a circle with its center at the origin, if the radius is "8 inches ± 1 %." (Note: This notation, frequently seen in drawings and applications, means here that the radius must be at least 7.92 inches long, and at most 8.08 inches long. We sometimes say that there is a "tolerance" of 1 % of the stated dimension.)

5-3. Parametric Representation.

In describing physical phenomena we customarily simplify matters; for example, a car on the road becomes a point on the line. In describing any motion it is convenient to say when, after some given instant, a particular event occurs. This is indicated by a value of the variable, t . If the motion takes place in two or three dimensions its analysis may be made easier by considering one dimension at a time. With a rectangular coordinate system we may then describe that part of the motion parallel to the x-axis (the x-component) by indicating how it alone changes with respect to time, say $x = f_1(t)$. Similarly we may have $y = f_2(t)$. Such a set of equations, in which the two components of the motion, that is, the values of the two variables x and y are given in terms of a third variable, t , is an example of what is called a parametric representation of the motion. It is interesting to note that the tracking of satellites is actually done in this way.

Example 1. Two students observe the motion of a ball rolling down a tilted plane. The plane has been coordinatized as indicated. In this illustration, as in many physical problems, the variable t , representing time elapsed since a given instant, is used as a parameter or auxiliary variable. The use of a parameter is often of great value in simplifying the presentation and solution of physical problems. In some problems it may be useful to use two, or even more, parameters.

One student finds that with suitable units he can describe the motion relative to the y-axis with the equation $y = 3t^2$.

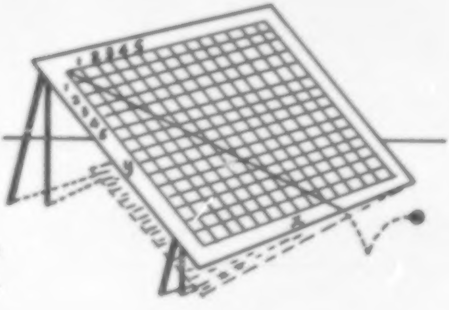


Figure 5-6

He may have come to this conclusion by noting, with the use of a stop-watch, the y-coordinates of the points on the lines parallel to the x-axis crossed by the rolling ball in successive seconds. The other student, using the lines parallel to the y-axis in a similar way, finds that he can describe the motion relative to the x-axis with the equation $x = 2t^2$. These are the parametric equations of the motion. If we want to express y in terms of x, we may eliminate t between these two equations and obtain $y = \frac{3}{2}x$. Since t is a measure of elapsed time it is nonnegative, hence x and y are also nonnegative. Therefore, the graph on the xy-plane will be a ray of the line whose equation may be written $y = \frac{3}{2}x$.

Example 2. A plane, flying at 120 miles per hour at an altitude of 5000 feet, drops a package to the ground. Assume that the package remains in one vertical plane as it falls, and, neglecting air resistance, determine its path to the ground.

Solution. We must assume certain conditions. If, at the moment of its release, the package is moving forward at 120 mph (= 176 ft. per sec.), then it will continue to do so at the same rate, whatever its vertical motion may be. Under the stated conditions we assume that its vertical motion is described by the formula $s = \frac{1}{2}gt^2$, where t represents the elapsed time in seconds, g is the gravitational acceleration in feet per second per second (which we shall approximate as 32), and s is the number of feet of free fall.

We now coordinatize the vertical plane, taking the point of release as the origin. The positive sense of the x-axis indicates forward motion, and the positive sense of the y-axis indicates downward motion.

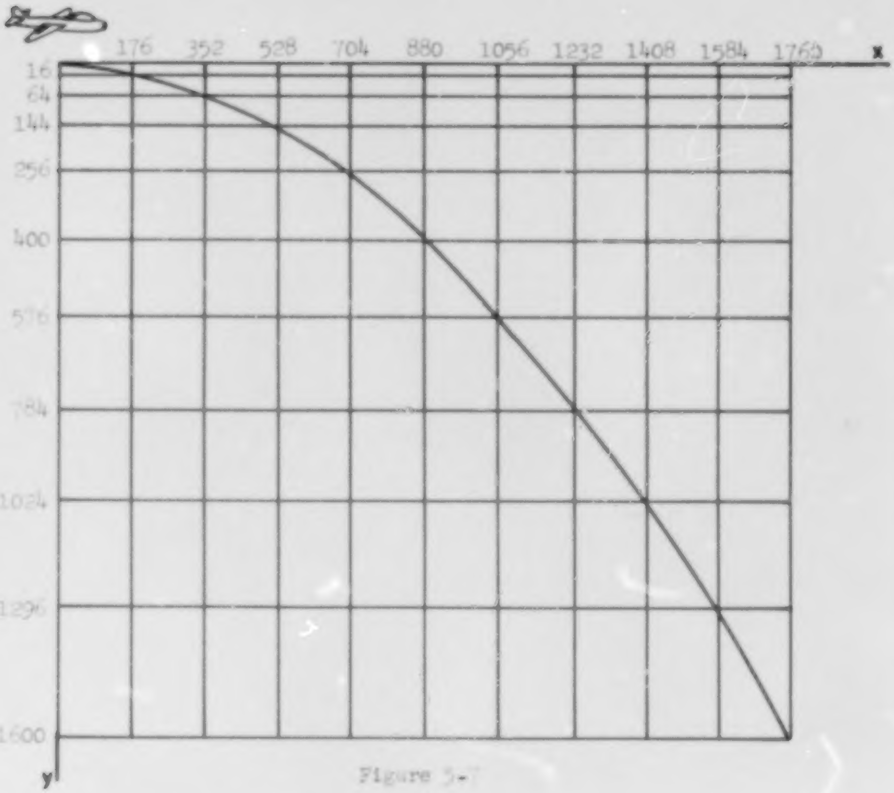


Figure 5-7

Note that the grid on which the locus is drawn has been presented in a non-standard way, to make the diagram easier to interpret. As the package moves forward in space the corresponding point on the graph moves right and crosses successive vertical lines in successive seconds. The vertical lines are equally spaced because the horizontal motion is uniform: $x = 176t$. As the package falls the corresponding point on the graph moves down on the page, crossing successive horizontal lines in successive seconds. The horizontal lines are not equally spaced because the vertical motion is not uniform, but accelerated. The spacing was determined by successive values of t in the formula $y = 16t^2$. The scale is the same on both axes, thus the diagram is not only a graph of our locus, but also a picture of the actual path.

If we had plotted points on a different grid, say the one to the right, in which the horizontal scale is different from the vertical scale, then the graph would still be an accurate representation of the relationships among the variables, but it would not be an accurate representation of the path. Since we use the word path here in a special way, we define it to be the set of positions actually occupied by a real object as it moves in real space. Clearly, a path may be represented by a curve in a great number of ways by different choices of coordinate systems.

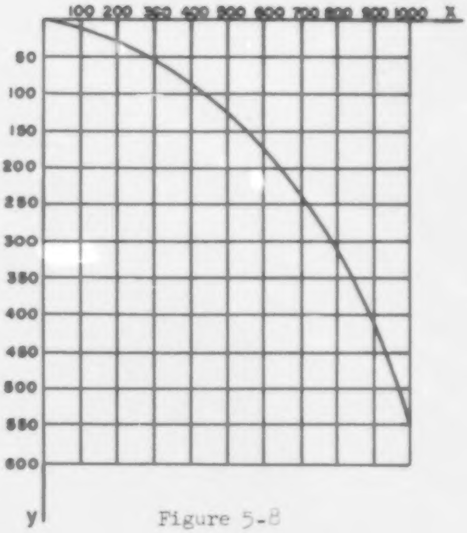


Figure 5-8

In many physical problems we are concerned with the relative positions of objects as they travel on their respective paths. If the bat is to hit the ball, it is not enough for their paths to cross, they must be at the crossing point at the same time. Ships' paths may cross safely, but a collision course would bring them to the same point at the same moment. The captains of two ships at sea are concerned with when and where the ships are closest to each other. When we must consider time and position along a path, we need some relationship involving these quantities. These are most readily presented in parametric form.

Exercises 5-3

1. Refer to Example 1 and make a chart like the one below, showing the x and y coordinates for integral values of t from $t = 0$ to $t = 10$.

t	0	1	2	3	4	5	6	7	8	9	10
x											
y											

2. Make a similar chart for Example 2 of this section.

3. Write parametric equations for the position of a point $P = (x,y)$ which starts on the y-axis and moves across the plane at the rate of 5 units a second, and remains always 2 units above the x-axis.

4. Write parametric equations for the position of a point $P = (x,y)$ which starts on the x-axis and moves uniformly on the plane at the rate of 2 units a second, and remains always 6 units to the left of the y-axis.

5. Write parametric equations for the position of a point $P = (x,y)$ which starts at the origin, goes through the point $(3,4)$ ten seconds later, and continues to move uniformly along line $\overleftrightarrow{OP}$ at that same rate across the plane. Find rectangular equations for its locus.

6. Write parametric equations for the position of a point $P = (x,y)$ which moves uniformly along a line across the plane, and takes 5 seconds to go from $(-6,1)$ to $(1,25)$.

7. Parametric equations for the path of a point $P = (x,y)$ are $x = t$, $y = t^2$, where t indicates time in seconds. Discuss the motion of the point in the first five seconds. Make an estimate, correct to the nearest .1 unit, of the distance traveled in that time.

8. A point $P = (x,y)$ travels along the line represented by $4x - 3y + 2 = 0$ at the uniform rate of 10 units per second and passes through $(1,2)$ when $t = 3$. Write parametric equations for its position at any time t. Find its position when $t = 0$; when $t = 10$.

9. A point $P = (x,y)$ travels along the line represented by $2x + 3y - 6 = 0$ at a uniform rate of 5 units per second and crosses the x-axis at the time $t = 0$. Write parametric equations for its position at any time t.

10. A point $P = (x,y)$ moves uniformly on a line across the plane. It goes through (a,b) at time t_0, and (c,d) at time t_1. Write parametric equations for its position at any time t.

11. A point is moving along the x-axis, its position at time t (sec) given by $x = \cos t$. Before you do any computation try to describe the way the point moves. The cosine function is frequently associated with angles and rotation, but there is no such motion here. We must now use the cosine as a particular real number function, whose values, for the domain $0 \leq x \leq 1.80$ are given in Table II. The heading "radian measure" for that table indicates the most frequent but by no means the only use for these trigonometric functions. Make a table for the positions of the point for the first 10 seconds, at one second intervals. How would you find the position of the point at the end of one minute? one hour?

12. The vertical position of a point is given by $y = 500 - 16t^2$ where y represents altitude in feet and t elapsed time in seconds. Before you do any computation try to describe the motion of the point. Do you know any physical interpretation of this motion? Make a table of the position of the point, at one second intervals, for the first 10 seconds.

13. Refer to the previous exercise, and answer the same questions for the relationship $y = 120 + 64t - 16t^2$.

14. Refer to Exercise 11, and answer the same questions for the relationship $x = 4 \sin 2t$.

15. Refer to Exercise 11, and answer the same questions for the relationship $x = 2 - \cos t$.

16. If the points of Exercises 11 and 15 were on the same x-axis, find a time and place at which they meet.

5-4. Parametric Equations of the Circle and the Ellipse.

In many physical situations an important role is played by a fixed reference point, such as a source of light or radiation or a magnetic pole. The associated phenomena, sometimes called focal or radial, can be described with polar coordinates or vectors. We should use the coordinate system and parameters which seem appropriate. When rotations are involved it is usually helpful to use as a parameter, θ , the measure of the angle of rotation from a fixed initial position.

Example 1. A point moves around a circle at constant speed. Find analytic conditions for its path.

Solution: Suppose, as in the diagram, the point starts from A and moves counter-clockwise. Its position at any point P is given by the rectangular coordinates (x,y) , or the equivalents $(r \cos \theta , r \sin \theta)$; that is,

$$\begin{cases} x = r \cos \theta , \\ y = r \sin \theta . \end{cases}$$

These are parametric equations for a circle.

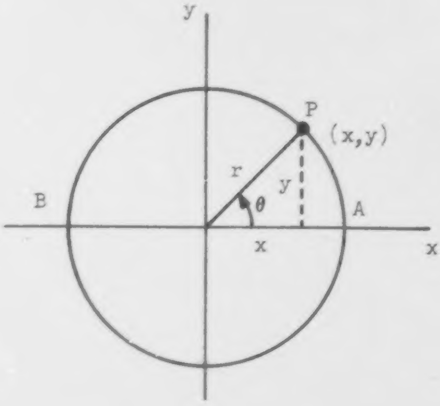


Figure 5-9

We may express the fact the point moves around the circle with constant speed by saying either that it moves along the circle at so many inches per second, or that the radius $\overline{OP}$ rotates about O at so many revolutions per minute. Of course, other units may be used. The first method of expression is important in mechanical problems involving, for example, gearing, belting, rimspeed, and so on. The second method of expressing constant speed, which concerns the amount of turning done in a unit of time, is significant in timing mechanisms such as are used in automatic washers, in electrical theory involving alternating current, which is related to the positions of a turning armature, and in the analysis of many other phenomena which are periodic, that is, which repeat in successive time intervals.

In this latter interpretation it is customary to use the Greek letter ω to represent the angular velocity, usually but not necessarily in terms of radians per unit time. Thus, if a wheel is turning at the rate of 300 revolutions per minute, it has an angular velocity of $(300)2\pi$ radians per minute, or 10π radians per second; that is, $\omega = 300(\text{rpm})$, or $\omega = 600\pi$ (radians/minute), or $\omega = 10\pi$ (radians/second).

If the point P has constant angular velocity ω, then its angular position θ is given by ωt. The parametric equations above become

$$\begin{cases} x = r\cos\omega t\ , \\ y = r\sin\omega t\ . \end{cases}$$

These are equations of the path of the point.

If we eliminate the parameter by squaring the members of each equation and adding the corresponding members of the new equations we obtain $x^2 + y^2 = r^2(\cos^2\omega t + \sin^2\omega t)$, or $x^2 + y^2 = r^2$. This represents the locus of the path in rectangular coordinates and no longer takes account of the position of the point at any particular instant.

Example 2. Two points travel on the same circle. They start at the same time from diametrically opposite positions and travel in opposite directions, the first at 2 rotations per second, the second at 3 rotations per second. Find analytic conditions for their paths, and the times and positions at which they coincide.

Solution. (Refer to Figure 5-9.) If the first point starts at $A = (r,0)$, and goes counterclockwise, its equations are

$$\begin{cases} x = r\cos 4\pi t\ , \\ y = r\sin 4\pi t\ . \end{cases}$$

If the second point starts at $B = (-r,0)$, and goes clockwise, its equations are

$$\begin{cases} x = r\cos(\pi - 6\pi t), \\ y = r\sin(\pi - 6\pi t). \end{cases}$$

If $t = 0$, the position of A is given by $(r\cos 0, r\sin 0)$; therefore $A = (r,0)$, as indicated. At the same time $(t = 0)$, the position of B is given by $(r\cos\pi, r\sin\pi)$; therefore $B = (-r,0)$, as indicated. As time elapses, the angle for the motion of A *increases*, while the angle for the motion of B *decreases*. As A and B rotate, only their angular positions are changing, and the rates of these angular displacements are 4π radians per second and -6π radians per second. At any instant the difference of these angular displacements is called their angular separation. It is customary to give this angular separation as the least angle between the respective radii to the points. Thus we use an angular separation of $\frac{\pi}{2}$ radians rather than 13.5π radians.

Since our two points start with an angular separation of π, their first meeting will occur when their angular displacements from their starting positions add to π; that is, when $4\pi t + 6\pi t = \pi$; $\therefore t = .1$ second. Successive meetings will occur *after* this when their *additional* angular displacements add to $2\pi, 4\pi, 6\pi, \ldots$, i.e., when $4\pi t + 6\pi t = 3\pi, 5\pi, 7\pi, \ldots$, i.e., when $t = .3, .5, .7, \ldots$. That is, they pass each other in .1 second, and every .2 second thereafter.

To find the corresponding positions, we need only substitute these values for t in the equations of motion. It is simplest to obtain first the successive angular positions $\theta_1, \theta_2, \ldots$, for their passing points.

If $t_1 = .1$, $\theta_1 = .4\pi = 72^\circ$.

If $t_2 = .3$, $\theta_2 = 1.2\pi = 216^\circ$.

If $t_3 = .5$, $\theta_3 = 2\pi = 360^\circ$.

The rectangular coordinates of these positions are given, say for $r = 10$, by $P_1 = (10\cos 72^\circ, 10\sin 72^\circ)$; $P_2 = (10\cos 216^\circ, 10\sin 216^\circ)$; $P_3 = (10\cos 360^\circ, 10\sin 360^\circ) \ldots$. These are equivalent to $P_1 = \big(10(.309), 10(.951)\big)$; $P_2 = \big(10(-.809), 10(-.588)\big)$; $P_3 = \big(10(1), 10(0)\big)$;

.... In usual rectangular form, rounded to hundredths, we have: $P_1 = (3.09, 9.51)$; $P_2 = (-8.09, -5.88)$; $P_3 = (10, 0)$; .... .

Example 3. (Refer to Example 2, above.) Suppose, in the previous exampl the points start as before but travel in the same direction, with the same rat as before. When and where do they pass?

Solution. The equations of motion are now:

$$\begin{cases} x = r\cos 4\pi t\ , \\ y = r\sin 4\pi t\ ; \end{cases} \quad \text{and} \quad \begin{cases} x = r\cos(\pi + 6\pi t)\ , \\ y = r\sin(\pi + 6\pi t)\ . \end{cases}$$

The meetings (or overtakings) will take place now when the difference of their angular displacements is 2π , 4π , 6π , ... The first meeting will take place when $\pi + 6\pi t - 4\pi t = 2\pi$; that is, when $t = .5$ sec. After this, successive meetings will occur when $\pi + 6\pi t - 4\pi t = 4\pi$, 6π , 8π , ... ; that is, when $t = 1.5$, 2.5 , 3.5 , ... To find the corresponding angular positions we proceed as in the previous problem and find $\theta_1 = 2\pi$, $\theta_2 = 6\pi$, etc.; that is, all overtakings will take place 1 second apart, at point A , starting at the end of the first half-second.

Example 4. A point is rotating uniformly on a circle of radius a , with its center at the point $(b,0)$. Find analytic conditions for its locus.

Solution. Suppose the uniform angular velocity, expressed in radians per second, is ω . From the hypothesis and the diagram, we have

$$\begin{cases} x = b + a\cos\theta\ , \\ y = a\sin\theta\ ; \end{cases} \qquad \begin{cases} x = b + a\cos\omega t\ , \\ y = a\sin\omega t\ . \end{cases}$$

These are parametric equations for the locus. The first equations are positional only, the second equations relate these positions to time and describe the path of the point.

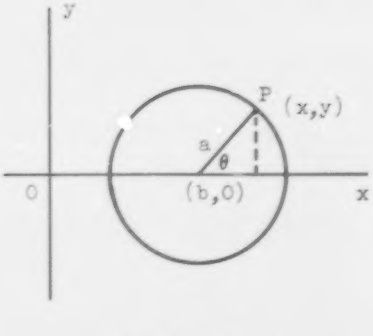


Figure 5-10

We may eliminate the parameters ω and t .

Since $$\frac{x-b}{a} = \cos\omega t\ ,\ \frac{y}{a} = \sin\omega t\ ,$$

therefore, $$\left(\frac{x - b}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1 \ ;$$

or $$(x - b)^2 + y^2 = a^2 \ .$$

This last equation is the one usually given in rectangular coordinates. It is an equation of the locus of the point and takes no account of its position at any particular moment.

The ellipse will be discussed in detail in Chapter 7, but we derive now its analytic representation in parametric form. We start with two concentric circles, the smallest that will enclose the ellipse, and the largest that the ellipse will enclose, as illustrated in Figure 5-11. Suppose their radii are a and b with $a > b$. We describe now a way in which a draftsman can locate as many points of the ellipse as he needs to draw a smooth curve through them. Draw any line through O , meeting the circles at A and B respectively. Through A and B the lines parallel to the y- and x-axes respectively will meet at point P of the ellipse. For all ϕ we have $x = d(O,C) = a \cos \phi$, and $y = d(C,P) = d(D,B) = b \sin \phi$.

The equations are

$$\begin{cases} x = a \cos \phi \ , \\ y = b \sin \phi \ . \end{cases}$$

We may eliminate ϕ as follows:

$$\frac{x}{a} = \cos \phi \ , \ \frac{y}{b} = \sin \phi \ ;$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi + \sin^2 \phi = 1 \ ,$$

or, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ ,$$

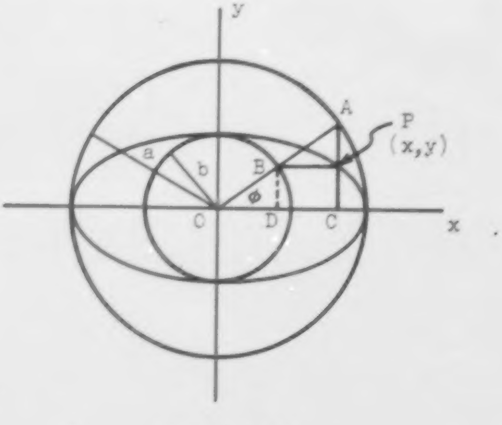


Figure 5-11

which is the usual equation of an ellipse in rectangular coordinates. Note that the parameter ϕ used here is not the angle between the positive part of the x-axis and the radius vector $\overrightarrow{OP}$ to the point P ; that is, it is not the angle used in representing P in polar coordinates.

It should be recognized that we may select a parameter in various ways to fit a variety of situations. There is never a unique way to do this, so it is inaccurate to refer to "the parametric equations of ...". Rather, we have "a parametric representation of ..." with the understanding that we have made the choices of constants and variables that best suit the hypothesis and our plan of approach to the solution.

Exercises 5-4

1. Write parametric equations for a circle of radius 10 and with center at the origin.

2. Write parametric equations for the path of a point around the circle of Exercise 1. Assume that it starts from the 3 o'clock position and rotates clockwise at the rate of 4 revolutions per second.

3. Write parametric equations for the path of a point at the end of the minute hand of a clock during one hour. Assume the length of the radius to be 6 inches and that the point starts from the 12 o'clock position to which we assign the numbers 0 and 60 . Use minutes as measures of time.

4. Write parametric equations for a circle with center at (4,0) and radius 3 .

5. Write parametric equations for a circle with center at (0,6) and radius 4 .

6. Write parametric equations for the path of a point moving around the circle of Exercise 4. Assume that it starts from its lowest point and moves clockwise at 2 rps .

7. Write parametric equations for the path of a point moving around the circle of Exercise 5. Assume that it starts from its highest point and moves counterclockwise at 3 rps .

Describe in words the motion of a point whose path has the parametric equations given below. Assume t denotes elapsed time in seconds.

8. $$\begin{cases} x = 4 \cos \pi t \ , \\ y = 4 \sin \pi t \ . \end{cases}$$

9. $$\begin{cases} x = 6 \cos (\pi t + \frac{\pi}{2}) \ , \\ y = 6 \sin (\pi t + \frac{\pi}{2} \ . \end{cases}$$

10. $$\begin{cases} x = 8 \cos (\pi - 3\pi t) \ , \\ y = 8 \sin (\pi - 3\pi t) \ . \end{cases}$$

11. $$\begin{cases} x = 10 \cos (\frac{3\pi}{2} + 10\pi t) \ , \\ y = 10 \sin (\frac{3\pi}{2} + 10\pi t) \ . \end{cases}$$

12. $$\begin{cases} x = 4 + \cos 6\pi t \,, \\ y = \sin 6\pi t \,. \end{cases}$$

13. $$\begin{cases} x = \cos 8\pi t \,, \\ y = -3 + \sin 8\pi t \,. \end{cases}$$

14. $$\begin{cases} x = 2 + \cos 12\pi t \,, \\ y = 5 + \sin 12\pi t \,. \end{cases}$$

15. $$\begin{cases} x = a + b \cos 2\pi t \,, \\ y = c + b \cos 2\pi t \,. \end{cases}$$

16. $$\begin{cases} x = p + q(\cos 2n\pi t - \alpha), \\ y = r + q(\cos 2n\pi t - \alpha). \end{cases}$$

17. The equations of motion of a point moving uniformly on a circular path are

$$\begin{cases} x = 6 \cos 4\pi t \,, \\ y = 6 \sin 4\pi t \,. \end{cases} \qquad (t \text{ in seconds})$$

(a) Describe its motion in words.

(b) Make a table showing the coordinates of the point at the times $t = 0 \,, .1 \,, .2 \,, \ldots \,, 1.0$ second.

(c) A second point travels on the same circle in the same direction at the same rate, and starts at the same time, but from the point on the y-axis above the origin. Write equations for its motion.

(d) A third point starts at the same time and place as the first point, but travels in the opposite direction at half its speed. Find equations of motion for this third point.

(e) Find the times and places at which the third point meets the first point, as was done in Examples 2 and 3.

(f) Find the times and places where the third point meets the second point.

18. Three bicyclists, A , B , C are equally spaced around a one mile circular track, (say at the 8 o'clock, 4 o'clock, and 12 o'clock positions, respectively). A and B , who go clockwise, can circle the track in 3 minutes and 4 minutes respectively. C , who travels counterclockwise, can circle the track in 5 minutes. They start at the same moment.

(a) Write equations of motion for their angular positions on the track at any time t after they start.

(b) Find and illustrate their positions at the end of each of the first 10 minutes.

(c) Determine the first 5 meetings; who meet; when, and where?

(d) When and where do all three meet, if ever?

19. A point starts at A (Figure 5-9) and moves counterclockwise at 2 rps. A second point starts at position P , which you are to find, and, moving clockwise at the same rate, passes the first point each time they cross the y-axis. Write the equations of motion for this second point.

20. Four points, P , Q , R , S are equally spaced around a circle (Figure 5-9), with P at the 3 o'clock position, Q at the 12 o'clock position, R at the 9 o'clock position, and S at the 6 o'clock position. P and Q move counterclockwise, R and S clockwise. They start simultaneously, and all meet for the first time 10 seconds later at the 10 o'clock position.

    (a) Write equations of motion for each point.

    (b) When and where will all four meet again?

5-5. Parametric Equations of the Cycloid.

A curve frequently encountered in physical applications is the *cycloid.* We introduce it in an example.

Example 1. A wheel of radius a feet rolls in a straight line down a flat road. Find analytic conditions for the path of a point P on the rim of the wheel.

Solution. Something--perhaps years of experience--suggests a parametric representation.

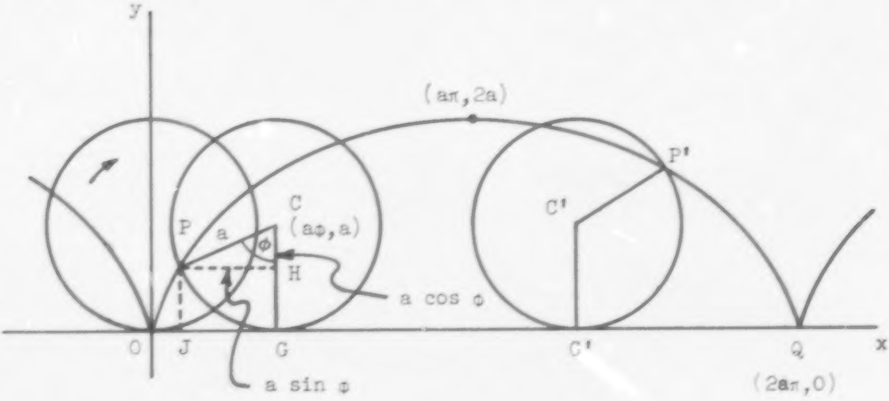


Figure 5-12

Let the line along which the wheel rolls be the x-axis, and let the origin be a point at which P touches the road. Let the positive direction on the x-axis be the direction in which the wheel is rolling. Finally, let ϕ be the radian measure of the angle through which the wheel has rotated since P touched the road, with ϕ positive when the center of the wheel has a positive abscissa. Since the wheel is rolling, not slipping, the length of $\overline{OG}$ is the same as the length of $\widehat{PG}$. The definition of radian measure gives this arc length as $a\phi$. Hence,

$$\begin{cases} x = d(O,J) = d(O,G) - d(P,H) = a\phi - a \sin \phi \, , \\ y = d(P,J) = d(C,G) - d(C,H) = a - a \cos \phi \, . \end{cases}$$

We rewrite these parametric equations of the cycloid

$$\begin{cases} x = a\phi - a \sin \phi \, , \\ y = a - a \cos \phi \, ; \end{cases} \quad \text{or} \quad \begin{cases} x = a(\phi - \sin \phi) \, , \\ y = a(1 - \cos \phi) \, . \end{cases} \tag{1}$$

If the wheel were rotating at the rate of ω radians per second, then $\phi = \omega t$ and Equations (1) become

$$\begin{cases} x = a\omega t - a \sin \omega t \, , \\ y = a - a \cos \omega t \, . \end{cases} \tag{2}$$

Exercises 5-5

1. A point $P = (x,y)$ on the rim of a wheel with a 2 inch diameter traces a cycloid as the wheel rolls along the x-axis. Write parametric equations for the locus of P. Find rectangular coordinates for P, correct to tenths, corresponding to values of θ from 0° to 360° at intervals of 30°. Make a careful drawing of the graph.

2. One arch of a cycloid will just fit inside a rectangle 6 units high. How wide is that rectangle? Choose suitable axes and then write parametric equations for the cycloid.

3. A wheel with a 6 inch diameter is rolling along a line, rotating 4 times per second.

   (a) Choose a suitable coordinate system and write parametric equations of the motion of a point $P = (x,y)$ on the rim.

   (b) Find rectangular coordinates for the positions of P at times $t = .1$, $.2$, $.3$, $.4$, $.5$.

   (c) Find the time and place at which P first reaches a high point on its path.

4. An automobile traveling along a straight and level road at 30 miles an hour has a wheel whose outer circumference is 66 inches.

   (a) Make an accurate scale drawing of one arch of the cycloid traced by a point on the circumference.

   (b) Choose a suitable coordinate system and write parametric equations for the motion of a point on the rim of the wheel. Use a minute as a unit of time and $3\frac{1}{7}$ as an approximate value for π.

Challenge Exercises for Sections 5-3, 5-4, 5-5

1. (Refer to Figure 5-12.) If, as in the case of a cycloid, we consider a wheel of radius a rolling down a straight flat road, we may consider the path of a point P not on the rim, but along a radius $\overline{CF}$, at a distance of b feet from the center. We distinguish two cases: $b > a$, and $b < a$. The locus in the first case is called a prolate cycloid, and in the second case a curtate cycloid. Figure 5-13 illustrates a case which leads to a prolate cycloid, whose parametric equations you are asked to find. A part of the graph is shown in Figure 5-14.

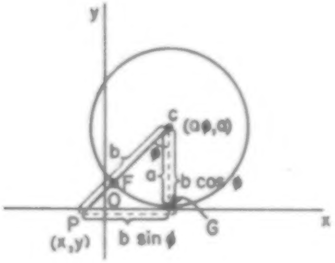


Figure 5-13

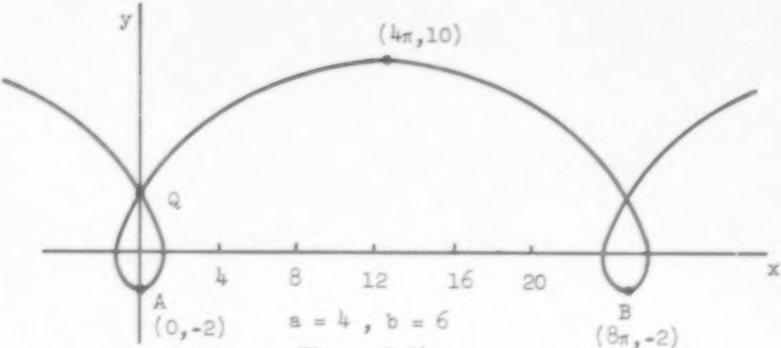


Figure 5-14

This figure illustrates a case in which $b = 1.5a$. (Can you find the ordinate of the point Q in which the graph cuts the y-axis?) The student is urged to consider the cases: $b = 2a$, $b = 10a$, and to draw some general conclusions.

2. The curtate cycloid. (Refer to Figures 5-13, 5-14.) Find the locus of a point P on the radius $\overline{CF}$ of a circle as the circle rolls along a line. $d(C,P) = b$; radius $= d(C,F) = a$, and $b < a$. Choose a suitable coordinate system and draw an arch of the graph of a curtate cycloid for the case $a = 6$, $b = 4$.

3. A circle of radius a rolls, without slipping, on the outside of a circle of radius b . Find an analytic representation of the locus of a point P on the outside circle.

Discussion: We illustrate the case $a < b$, and suggest these relations: length of $\widehat{AB}$ = length of $\widehat{PB}$, $\therefore a\phi = b\theta$.
$C = \left((a + b) \cos \theta , (a + b) \sin \theta\right)$;
the sum of the measures of θ , ϕ , ψ is $\frac{\pi}{2}$ or 90° ;
$d(P,D) = a \sin \psi$; $d(C,D) = a \cos \psi$.
We urge the student to experiment with the special cases $a = b$, $a = \frac{1}{2}b$, $a = \frac{1}{3}b$. Such curves are called epicycloids and have applications in astronomy and in mechanical engineering.

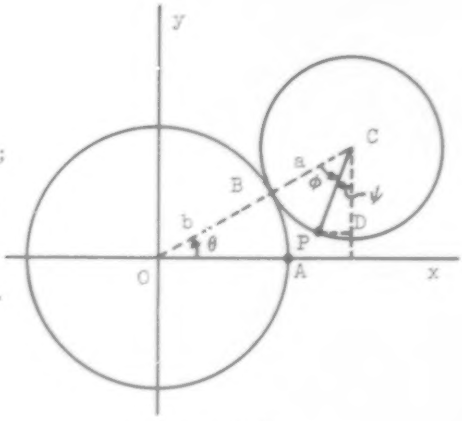


Figure 5-15

4. (Refer to the previous problem.) A circle of radius a rolls, without slipping, on the inside of a circle of radius b $(a < b)$. Find analytic representations of the path of a point P on the circumference of the inside circle. Such a path is called a hypocycloid. The student is urged to experiment with the special cases $a = \frac{1}{4}b$, $a = \frac{1}{3}b$, $a = \frac{1}{2}b$. In both this and the previous exercise the student is challenged to answer this question without performing the experiment: If $a = \frac{1}{2}b$, and we make a complete circuit, how many times has the smaller circle rotated on its own axis?

5. A circle of radius a has as center $C = (0,a)$. A chord is drawn through any point $D = (x_1,y_1)$ of the circle and extended to meet, at Q, the tangent to the circle at A, the end of the diameter from O. $\overline{QR}$ is drawn parallel to $\overline{AO}$, and a line is drawn from D parallel to $\overline{AQ}$, and intersecting $\overline{QR}$ at $P = (x,y)$. Find equations of the locus of P as the point D moves on the circle. Sketch the locus. (This curve, called the witch of Agnesi, was studied and named by a mathematician of the eighteenth century, Maria Gaetana Agnesi.)

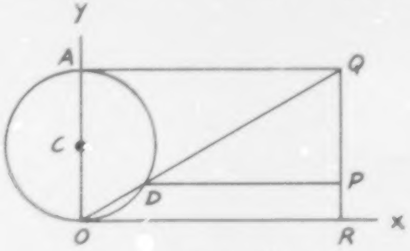


Figure 5-16

6. Find an equation of the locus of a point which moves so that the sum of the squares of its distances from two fixed points is a constant, which we call $2a^2$. Describe and sketch the locus.

7. Find an equation of the locus of a point which moves so that the sum of the squares of its distances from the vertices of a square is constant. Describe the locus.

8. Find an equation of the locus of a point which moves so that the sum of the squares of its distances from the lines containing the sides of a square is constant.

9. A line drawn parallel to the side $\overline{AB}$ of a triangle ABC, meets $\overline{AC}$ in D, $\overline{BC}$ in E. The lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{BD}$ meet at P. Find an equation of the locus consisting of all such points P. (Hint: Let $\overleftrightarrow{AB}$ be the x-axis and let $C = (0,c)$, where $c > 0$. Introduce, as a parameter, t, the distance between $\overline{DE}$ and the x-axis.)

10. Let O and Q be distinct points. Let L be a line through O and let P be the foot of the perpendicular to L through Q . What is the locus of P as L rotates around O ? (Hint: Use the slope of L as an auxiliary variable. Remember that some lines don't have slopes. Does Q lie on the locus?)

11. A circle of radius a has its diameter OCA along the polar axis. From O a chord $\overline{OR}$ is drawn and extended to meet, at S, the tangent to the circle at A . Find equations of the locus of P , a point on $\overrightarrow{OS}$ such that d(P,S) = d(OR) . Make a sketch of the graph. (This locus is a cissoid, a curve studied by the Greek mathematician Diocles, who lived a century or so after Euclid. You may learn something more about it when you study inversion later.

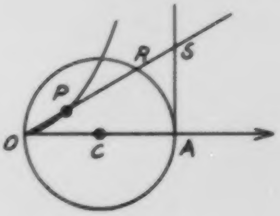


Figure 5-17

12. A fixed line $\overleftrightarrow{BC}$ is perpendicular to the polar axis at point A , a units from the pole. A line is drawn through O meeting $\overleftrightarrow{BC}$ at R . A fixed length ℓ is marked off from R on this line in both directions locating the points P and P' . Find an equation in polar coordinates for the locus of P and P' . (This curve, called a conchoid, was studied by the Greek mathematician Nicomedes about two centuries B.C. It can be used in the trisection of an angle. Try to discover how.)

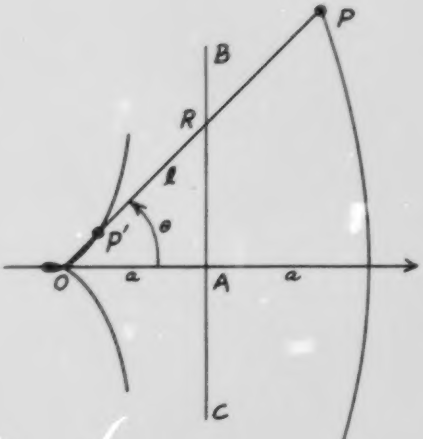


Figure 5-18

13. Involute of the circle. A string of no thickness is wrapped around a fixed circle; the end of the string is at A. We unwrap the string, keeping it taut, and tangent to the circle. ($\overline{PT}$ is tangent to the circle, and $d(P,T) =$ length of $\widehat{AT}$). Find analytic conditions for the graph of P. This graph is called the involute of the circle. Try to generalize this idea, and sketch involutes for an ellipse, a parabola, ... Does every curve have an involute? Make some mechanical models with which you can draw involutes. Draw the involute of a square.

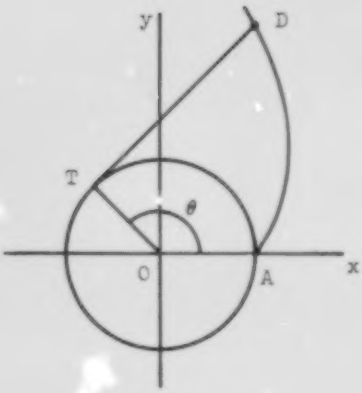


Figure 5-19

14. Suppose a fixed circle with radius a is internally tangent to a circle with radius b $(b > a)$. Find parametric equations for the locus of a point P on the outer circle as the outer circle rolls around the inner circle without slipping.

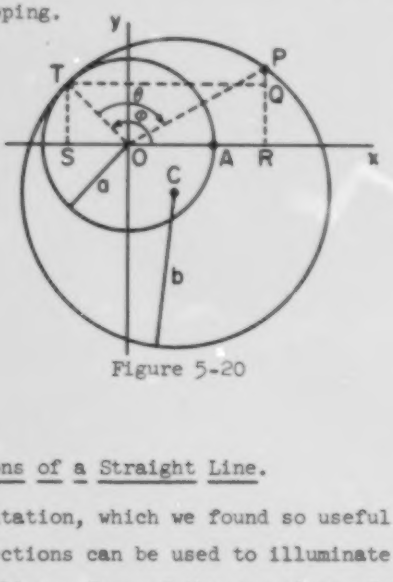


Figure 5-20

5-6. Parametric Equations of a Straight Line.

Parametric representation, which we found so useful in the complicated cases of the previous sections can be used to illuminate and extend the discussion of the straight line. Some of the exercises of Section 5-2 have already introduced you to the ideas and methods we examine now in more detail. The foundations for this discussion have already been developed in Chapter 2, particularly in Section 3, where we find these equations:

(1)
$$\begin{cases} x = x_0 + t(x_1 - x_0) , \\ y = y_0 + t(y_1 - y_0) . \end{cases}$$

We recognize that the quantities $x_1 - x_0$ and $y_1 - y_0$ are direction numbers obtained from the coordinates of the point $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. Therefore, we represent them respectively by ℓ and m, and rewrite Equations (1) as

$$(2) \qquad \begin{cases} x = x_0 + \ell t , \\ y = y_0 + mt . \end{cases}$$

We recognize that t is a parameter, and that these equations are parametric equations of the line through the points P_0 and P_1, which we assume to be distinct.

If $x_1 = x_0$, then $y_1 \neq y_0$, and (2) takes the form

$$\begin{cases} x = x_0 , \\ y = y_0 + mt . \end{cases}$$

(What is the geometric version of this hypothesis and conclusion?)

If $y_1 = y_0$, then $x_1 \neq x_0$, and (2) takes the form

$$\begin{cases} x = x_0 + \ell t , \\ y = y_0 . \end{cases}$$

(What is the geometric version of this hypothesis and conclusion?)

Example 1. Find a parametric representation of the line through (2,0) and (-4,3).

Solution: We can choose either point as P_0. If $P_0 = (2,0)$ then $x_1 - x_0 = -6$, $y_1 - y_0 = 3$ and we get the representation

$$\begin{cases} x = 2 - 6t , \\ y = 0 + 3t . \end{cases}$$

The other choice for P_0 leads to the representation

$$\begin{cases} x = -4 + 6t , \\ y = 3 - 3t . \end{cases}$$

A parametric representation of a line sets up a one-to-one correspondence between the real numbers and the points on a line in the plane. We illustrate below the correspondences established by the parametric representations we found for the line in Example 1.

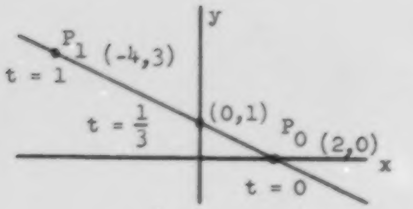


Figure 5-21a

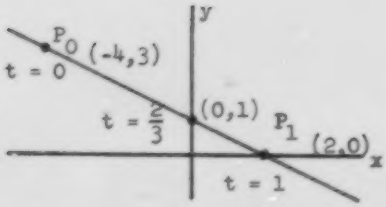


Figure 5-21b

Example 2. Find the intersection of the line through $(4,2)$ and $(2,-4)$ and the line through $(-3,-1)$ and $(-4,2)$.

Solution: The lines may be represented parametrically as follows:

$$L_1: \begin{cases} x = 4 - 2s, \\ y = 2 - 6s. \end{cases} \qquad L_2: \begin{cases} x = -3 - t, \\ y = -1 + 3t. \end{cases}$$

We wish to find all points which lie on both lines. Now the point (x,y) lies on both lines if and only if there exist values s_0 and t_0 of s and t such that

$$x = 4 - 2s_0 = -3 - t_0,$$
$$y = 2 - 6s_0 = -1 + 3t_0.$$

All such values of s and t can be found by solving simultaneously the equations

$$4 - 2s = -3 - t,$$
$$2 - 6s = -1 + 3t.$$

The only solutions are $s = 2$, $t = -3$. Substituting these in either pair of parametric equations, we find that the only point of intersection is $(0,-10)$.

It would have been quite correct to use the same letter for the parameter in the parametric representations of L_1 and L_2. However, this would have led to difficulties later in the problem. Do you see why? Can you find another method of getting around the difficulties?

In previous sections of this chapter we related the parameter t to elapsed time. In such cases the parametric equations gave us equations of motion of the point P. The graph of these equations was directly related to the path of the point. Example 3 shows how this approach can be used for the line.

Example 3. A ball is rolling along a level surface in a straight line with constant velocity. The surface is provided with a Cartesian coordinate system with the foot as the unit of length. At 10:00 a.m. the ball is at $(4,2)$ while one second later it is at $(2,-4)$. A second ball, also rolling along the level surface in a straight line with a constant velocity, is at $(-4,2)$ at 10:00 a.m., at $(-3,-1)$ one second later. We ask whether the two balls will collide. In other words, we want to know not whether their paths intersect but whether, if they do, the two balls are at any point of intersection at the same time. We assume, in order to simplify the problem, that the balls have zero radii and will collide only if their centers coincide.

Solution. The path of the first ball is represented by the equations

$$\begin{cases} x = 4 - 2s \,, \\ y = 2 - 6s \,. \end{cases}$$

Further, if s is the number of seconds which have elapsed since 10:00 a.m., the equations also tell us where the ball is at any time. For if we set $s = 0$ (10:00:00 a.m.) we get $x = 4$ and $y = 2$, while if we set $s = 1$ (10:00:01 a.m.) we get $x = 2$ and $y = -4$. Further, in s seconds starting at 10:00:00 a.m., an object whose motion was represented by these equations would travel

$$\sqrt{(x-4)^2 + (y-2)^2} = \sqrt{4 + 36}\, s = 2\sqrt{10}\, s$$

feet. Thus the distance travelled is a constant multiple of the time taken and the speed is constant. Similarly, the motion of the second ball is described by the equations

$$\begin{cases} x = -4 + t \,, \\ y = 2 - 3t \,. \end{cases}$$

Our problem is to find out whether the abscissas of the positions of the two balls, and the ordinates, are ever simultaneously $(s = t)$ equal. In other words we ask whether the system of equations

$$\begin{cases} 4 - 2t = -4 + t , \\ 2 - 6t = 2 - 3t . \end{cases}$$

has a solution. Clearly not, since this pair is equivalent to the pair

$$\begin{cases} 3t = 8 , \\ 3t = 0 . \end{cases}$$

Thus the balls do not collide.

If direction cosines are used in a parametric representation of a line, the parameter t has an interesting interpretation. Since

$$d(P_0,P) = \sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{\lambda^2t^2 + \mu^2t^2} = |t| ,$$

the absolute value of the parameter is the distance of the corresponding point P from P_0 .

__Example 4.__ Find, on the line through $P_0 = (1,5)$ and $P_1 = (5,8)$, two points which are 3 units distant from P_0 .

__Solution.__ Direction numbers for $\overleftrightarrow{P_0P_1}$ are $(4,3)$, and direction cosines can be taken as $\left(\frac{4}{5} , \frac{3}{5}\right)$. We may then write parametric equations for $\overleftrightarrow{P_0P_1}$ in terms of direction cosines as

$$\begin{cases} x = 1 + \frac{4}{5}t , \\ y = 5 + \frac{3}{5}t . \end{cases}$$

The substitution $t = \pm 3$ gives the coordinates of both points, $\left(1 \pm \frac{12}{5} , 5 \pm \frac{9}{5}\right)$, or $(3.4,6.8)$ and $(-1.4,3.2)$

Exercises 5-6

1. Find two parametric representations for each line through one of the following pairs of points, using each pair in both possible orders.

   (a) $(5,-1)$, $(2,3)$
   (b) $(0,0)$, $(4,1)$
   (c) $(2,-3)$, $(2,3)$
   (d) $(-1,4)$, $(-6,4)$
   (e) $(1,1)$, $(2,2)$
   (f) $(-1,-1)$, $(1,1)$
   (g) $(1,0)$, $(0,1)$
   (h) $(2,-2)$, $(-2,2)$

2. Draw the graph of each of the lines in Exercise 1, plotting, on each, the points corresponding to the values -1 , 0 , 1 , and 2 of the parameter.

3. Find the intersection of each of the following pairs of lines. When the lines do not intersect, what do you notice about their equations?

   (a) $\begin{cases} x = 5 + s \\ y = 2 - s \end{cases}$ $\qquad$ $\begin{cases} x = 4 - 2t \\ y = -6 + 3t \end{cases}$

   (b) $\begin{cases} x = 2 - 3s \\ y = 1 + 2s \end{cases}$ $\qquad$ $\begin{cases} x = 4 + 6t \\ y = -5 - 4t \end{cases}$

   (c) $\begin{cases} x = -3 + s \\ y = 2 - 3s \end{cases}$ $\qquad$ $\begin{cases} x = -2 - t \\ y = -1 + 3t \end{cases}$

4. Find a pair of parametric equations for the line L with equation $2x - 3y + 1 = 0$.

5. Let L have the parametric equations

$$\begin{cases} x = x_0 + \ell t \ , \\ y = y_0 + mt \ . \end{cases}$$

   Let $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ be the points on L given by $t = t_1$, and $t = t_2$, respectively. Prove that $d(P_1,P_2) = \sqrt{\ell^2 + m^2}\,|t_2 - t_1|$.

6. A ball is rolling on a level floor along the line through $(16,2)$ and $(4,7)$ and in the direction from the first point towards the second. (The unit of length is the foot.) Its speed is 26 feet per second. Find parametric equations for its motion, measuring time from the instant when it is at $(16,2)$.

7. Let S be a set of points in a plane. A point P is sometimes called a center of S if S is symmetric about P. A parametric representation of a line may be used to prove that a point is a center of a set of points. Let S be the circle with equation $x^2 + y^2 = 4$. Any line through the origin has a parametric representation $x = \lambda t$, $y = \mu t$, with $\lambda^2 + \mu^2 = 1$. Substituting these expressions for x and y in the equation of the circle we get

$$\lambda^2 t^2 + \mu^2 t^2 = 4,$$

or
$$t^2 = 4.$$

Thus
$$t = \pm 2.$$

Since the answer is independent of λ and μ, every line through the origin meets the circle in the points given by $t = -2$ and $t = 2$. These are equidistant from the origin.

(a) Show that the origin is a center for $b^2x^2 + a^2y^2 = a^2b^2$.

(b) Show that the origin is a center for $y = ax^3$. (Discuss the case when $a > 0$ and the case when $a < 0$.)

(c) Show that the origin is a center for $y = \dfrac{x^3}{x^2 - 1}$.

8. A set S of points in a plane is called bounded if there is a rectangle which contains S. Prove that a bounded set in a plane has at most one center. Is this also true for unbounded sets?

9. Find, on the line through $P_0 = (1,5)$ and $P_1 = (5,8)$, two points at unit distance from P_1.

10. Find, on the line through $A = (-3,5)$ and $B = (0,9)$, two points P and Q such that $d(B,P) = d(B,Q) = 5d(A,B)$.

5-7. Summary.

We have investigated the relations between certain geometric and algebraic entities. The geometric objects were sets of points not, as we have said, given to us in a basket but determined by certain conditions or descriptions. The corresponding algebraic expressions were statements of equality or inequality. The relations between them were approached through a coordinatization of the "space" in which the sets were presented to us. Then our knowledge and ingenuity and experience led us to an algebraic description of the set, in the terminology of our coordinate system.

We have shown this process in detail in a number of situations. We have applied parametric representation in situations involving angular displacement and motions along a circle or line. If a set of points has any special properties or geometric appearance, how is this reflected in its analytic representation? If, for example, the set of points is symmetric in any way, could we tell that from its equation? If, on the other hand, some analytic representation shows a particular algebraic property, what is the geometric counterpart? What would be the geometric effect of imposing certain restrictions on the domain or range of the variables that appear in the analytic representations?

In our next chapter we will investigate in detail many such relations between curves and their analytic representations.

Review Exercises

1. We describe certain sets of points. You are asked to give an analytic description of each.

   (a) All points equidistant from the x- and y-axes.
   (b) All points equidistant from the points $A = (5,0)$ and $B = (11,0)$.
   (c) All points equidistant from $A = (5,0)$ and $C = (5,8)$.
   (d) All points equidistant from $C = (5,8)$ and $B = (11,0)$.
   (e) All points at distance 3 from $C = (5,8)$.
   (f) All points at distance 3 from the line $x = 5$.
   (g) All points at distance 3 from the line $y = -2$.
   (h) All points at distance 3 from the line $3x - 4y + 7 = 0$.
   (i) All points at distance h from the line $x = k$.
   (j) All points at distance p from the line $y = q$.
   (k) All points at distance d from the line $ax + by + c = 0$.
   (l) All points twice as far from $A = (5,0)$ as from $B = (11,0)$.
   (m) All points equidistant from the point $C = (5,8)$ and the x-axis.
   (n) All points equidistant from the point $A = (5,0)$ and the line $x = 1$.
   (o) All points equidistant from the point $D = (5,3)$ and the line $3x - 4y + 7 = 0$.
   (p) All points equidistant from the line $ax + by + c = 0$ and the point $P = (r,s)$ not on that line.

2. If $A = (-3,1)$, $B = (5,3)$, $C = (1,5)$, find an analytic representation of

(a) $\overleftrightarrow{AB}$
(b) $\overrightarrow{AB}$
(c) $\overline{AB}$
(d) $\overleftrightarrow{BC}$
(e) $\overrightarrow{BC}$
(f) $\overline{BC}$
(g) $\overleftrightarrow{CA}$
(h) $\overrightarrow{CA}$
(i) $\overline{CA}$

(j) the interior of $\angle ABC$.
(k) the interior of $\angle BCA$.
(l) the interior of $\angle CAB$.
(m) the interior of $\triangle ABC$.
(n) the line through A and parallel to $\overleftrightarrow{BC}$.
(o) the line through B and parallel to $\overleftrightarrow{CA}$.
(p) the line through C and parallel to $\overleftrightarrow{AB}$.
(q) the line containing altitude $\overline{AD}$ of $\triangle ABC$.
(r) the line containing altitude $\overline{BE}$ of $\triangle ABC$.
(s) the line containing altitude $\overline{CF}$ of $\triangle ABC$.
(t) the line containing the median of $\triangle ABC$ through A .
(u) the line containing the median of $\triangle ABC$ through B .
(v) the line containing the median of $\triangle ABC$ through C .
(w) the pair of lines through A and parallel to the axes.
(x) the perpendicular bisector of $\overline{AB}$.
(y) the perpendicular bisector of $\overline{BC}$.
(z) the circle containing A , B , and C .

3. The following expressions are analytic descriptions of certain sets. You are asked to describe each set in words, giving its name, its location on the plane, and any special geometric properties it may have. Sketch the graph of each.

(a) $\frac{x}{3} + \frac{y}{5} = 1$
(b) $\frac{x}{3} + \frac{y}{5} = 5$
(c) $x^2 = 16$
(d) $x^2 + y^2 = 16$
(e) $x^2 + 9y^2 = 16$
(f) $x^2 - 9y^2 = 16$
(g) $x^2 - 9y = 16$
(h) $9y - x^2 = 16$
(i) $y^2 - 9x = 16$
(j) $|x - 3| = 5$
(k) $|x + 5| < 4$
(l) $|x - a| \leq b$
(m) $xy = 0$
(n) $(x - 1)(y + 2) = 0$
(o) $x^2 - 3x - 10 = 0$
(p) $x < y$
(q) $x^2 < y^2$
(r) $x < x^2$

4. Give verbal descriptions of each of the sets described analytically with polar coordinates below. Give its name if available, its location on the plane, and any special geometric properties it may have.

(a) $r^2 = 9$

(b) $r^2 < 9$

(c) $r < 3$

(d) $r > 3$

(e) $\theta = 2$

(f) $\theta < \frac{\pi}{2}$

(g) $r = 2\theta$

(h) $r < \theta$

(i) $|\theta - 2| = .1$

(j) $|r - 5| < .1$

(k) $r = \frac{6}{\sin \theta}$

(l) $r = \frac{-3}{\cos \theta}$

(m) $r = \frac{-2}{\cos \theta}$

(n) $r = \frac{5}{\cos \theta}$

(o) $r = \frac{1}{\cos(\theta + \frac{\pi}{4})}$

(p) $r = \frac{4}{\sin(\theta - \frac{\pi}{4})}$

(q) $r = \frac{a}{\sin(\theta - b)}$

(r) $r > \frac{1}{\sin \theta}$

(s) $r < \frac{2}{\cos \theta}$

(t) $r = 0$.

5. Write the related polar equation or inequality for each part of Exercise 4 above.

6. Eliminate the parameter in each pair of parametric equations below.

(a) $\begin{cases} x = 1 + t \\ y = 1 + t^2 \end{cases}$

(b) $\begin{cases} x = 2t \\ y = t + 2 \end{cases}$

(c) $\begin{cases} x = \frac{1}{t + 1} \\ y = \frac{1}{2t + 1} \end{cases}$

(d) $\begin{cases} x = t^2 + t \\ y = t^3 + t^2 \end{cases}$

(e) $\begin{cases} x = t + \frac{1}{t} \\ y = t^2 + \frac{1}{t^2} \end{cases}$

(f) $\begin{cases} x = 3 \sin t \\ y = r \cos t \end{cases}$

(g) $\begin{cases} x = 2 + 3 \cos t \\ y = 4 - 5 \sin t \end{cases}$

(h) $\begin{cases} x = 2 \sin t \\ y = \sin 2t \end{cases}$

(i) $\begin{cases} x = \frac{1}{\sin t} \\ y = \frac{1}{\cos t} \end{cases}$

(j) $\begin{cases} x = \sin 2t \\ y = \sin \frac{1}{2}t \end{cases}$

7. A point moves on a line from $A = (3,7)$ through $B = (0,3)$ at the rate of 1 linear unit per second. Write parametric equations for its path, using seconds as units for the parameter t.

8. A point moves on a line from the origin through point $C = (7,24)$ at the rate of 5 linear units per second. Write parametric equations for its path, using minutes as units for the parameter t.

9. A point A moves along a line with parametric equations for its path: $\begin{cases} x = -1 + 3t, \\ y = 3 - t \end{cases}$. Point B moves along a line with parametric equations for its path: $\begin{cases} x = 5 - 2t, \\ y = 11 + t. \end{cases}$ Find $d(A,B)$ when $t = 3$, and when $t = 5$.

10. The path of P_1 has equations $\begin{cases} x = x_1 + \ell_1 t, \\ y = y_1 + m_1 t. \end{cases}$

The path of P_2 has equations $\begin{cases} x = x_2 + \ell_2 t, \\ y = y_2 + m_2 t. \end{cases}$

Express $d(P_1,P_2)$ when $t = 2$, in terms of the constants in these equations.

11. Write parametric equations for each path of a point around the rim of a clock if the path has the following description (assume unit radius):

(a) Starts at 12 o'clock position, and moves counterclockwise at 3 rps (revolutions per second).
(b) Starts at 6 o'clock position and moves clockwise at 2 rps.
(c) Starts at 4 o'clock position and moves counterclockwise at 1 rps.
(d) Starts at 9 o'clock position and moves clockwise at 4 rps.
(e) Starts at 8 o'clock position and moves counterclockwise at $\frac{1}{2}$ rps.

12. Find the time and place of the first meeting, assuming a simultaneous start of the points described in Exercise 11:

(a) a and b
(b) a and c
(c) a and d
(d) a and e
(e) b and c
(f) b and d
(g) b and e
(h) c and d
(i) c and e
(j) d and e

13. A point is rotating counterclockwise at 2 rps at a distance 3 from the point $(4,5)$. Find analytic conditions for its path.

14. A point is rotating clockwise at 1 rps at a distance of 2 from the point $(-1,0)$. Find analytic conditions for its path.

15. We give analytic descriptions of the paths of certain points around the rim of a clock. You are asked to describe these parts in words. Assume t measured in minutes.

(a) $\begin{cases} x = 4\cos 4\pi t \\ y = 4\sin 4\pi t \end{cases}$

(b) $\begin{cases} x = 6\cos(\frac{\pi}{2} + 6\pi t) \\ y = 6\sin(\frac{\pi}{2} + 6\pi t) \end{cases}$

(c) $\begin{cases} x = 10\cos(\pi - 10\pi t) \\ y = 10\sin(\pi - 10\pi t) \end{cases}$

(d) $\begin{cases} x = 8\cos(4\pi t + \pi) \\ y = 8\sin(4\pi t + \pi) \end{cases}$

(e) $\begin{cases} x = 2\sin 2\pi t \\ y = 2\cos 2\pi t \end{cases}$

16. Find parametric representations for the ellipses described below:

(a) center at the origin, major axis 10 along the x-axis, minor axis 6.

(b) center at the origin, x-intercepts ± 3, y-intercepts ± 4.

(c) major axis horizontal, and the ellipse will just fit between the circles $x^2 + y^2 = 5$ and $x^2 + y^2 = 6$.

17. A wheel with radius 12 inches, turning at the rate of 3 rps, is rolling down a straight, level road. Assume a coordinate system as usual and write parametric equations for

(a) a point P on its rim;

(b) a point Q, six inches in from the rim. (A challenge problem.)

Chapter 6

CURVE SKETCHING AND LOCUS PROBLEMS

6-1. Introduction

We have by this time made a beginning in the discussion of sets of points and their analytic descriptions. We have introduced and used various coordinate systems. We have used parametric representations, finding them particularly useful in physical applications involving rotation or other motion, and in locating positions on a path. Now we investigate more of the details and try to develop more competence (and confidence) in this powerful language of analytic geometry.

6-2. General Principles

The study of analytic geometry has two major concerns. One of these is the relation of geometry to algebra; the other is the relation of algebra to geometry. We must, therefore, consider two basic situations.

A. We are given a set of points. What would be a good analytic representation of that set? If we had two sets of points how would their geometric relationships be revealed in their analytic representations? (geometry to algebra.)

B. We are given an analytic representation of a set of points. What can we now say about the geometric properties of that set? If we had analytic representations of two sets, how could we use these to reveal and develop their geometric relationships? (algebra to geometry.)

In the first situation we must distinguish immediately between the cases we shall treat in this text and those we leave for later work. If a set of points comes to us, say, from a chart of the results of an experiment or a curve drawn by an automatic recording device, it might be useful to have a simple analytic representation of that set. We do not treat such matters in this book, although they have important applications in science, and are the subject of much current mathematical research.

The sets of points with which we shall concern ourselves must come already structured by some geometric condition or property. Our task will be to translate this condition into analytic terms through our choice of coordinate system and mode of algebraic or trigonometric representation. For example we may be interested in the set of all points equidistant from two given points. What type of coordinate system is best suited to describe this situation? Can we simplify the description by a wise choice of axes and units?

On the other hand suppose we meet the expression $2x + 3y + 5 \geq 0$. What set of points does it describe? Is it a configuration we can visualize? What are its properties?

In this second situation the variables come to us already named, and the context and notation usually indicate the type of coordinate system and the choices of axes and units. The analytic representation may exhibit some special algebraic or trigonometric properties which we expect to see reflected in certain geometric properties of the corresponding graph. We do not define the general term, "property", but illustrate and comment on those we shall consider.

Example 1. Discuss the equation $y = \sin x$ and its graph.

Discussion: We assume that the domain of x is the set of real numbers and note immediately that, whatever the value of x, we always have $|y| \leq 1$. If a graph of this equation were drawn on the usual rectangular coordinate grid the geometric interpretation of this statement is that the entire graph is contained in a strip two units wide, centered on the x-axis, and of infinite length to right and left. We sometimes describe such restrictions on the graph by saying it is bounded above and below, but not at the sides. Any comment indicating what regions of the plane may or may not be occupied by a graph is part of the discussion of what is called the extent of the graph.

We note also from the given relationship, that for each value of x there is a unique value of y, but not vice versa. That is, y is expressed as a function of x, but x is not a function of y. The geometric version of this comment is that, if the graph were drawn on the usual rectangular coordinate grid, each line parallel to the y-axis would intersect the graph exactly once. What can you say about intersections of the graph with lines parallel to the x-axis?

We note also that, since $\sin(x + 2n\pi) = \sin x$ for integral values of n, the y values will repeat endlessly through the range $-1 \leq y \leq 1$. We say in this case that y is a periodic function of x. If, in general, $y = f(x)$ so that, for some fixed $p \neq 0$ and for all x, $f(x + p) = f(x)$, then we say that y is a periodic function of x. In that case

$$f(x + 2p) = f((x + p) + p) = f(x + p) = f(x) .$$

Therefore, for such functions, $f(x + np) = f(x)$ for integral values of n. If $p > 0$ and there is no smaller positive number which satisfies the requirement: $f(x + p) = f(x)$ for all x, then we say that $f(x)$ is a periodic function of x, of period p.

Specifically, $y = \sin x$ is a periodic function of x of period 2π. What are the periods of the periodic functions, $y = \cos x$ and $y = \tan x$? Note that it is the function whi[illegible]eriodic, not the graph. A particular function may have quite different loo[illegible]g graphs, depending on our choices of coordinate systems. The periodicity of a function may be more readily seen in some graphs than in others. The graph in Figure 6-1 can be interpreted to give the same information about $y = \sin x$ as is given when we say that y is a bounded periodic function of x of period 2π. What other information about the function can be inferred from the graph?

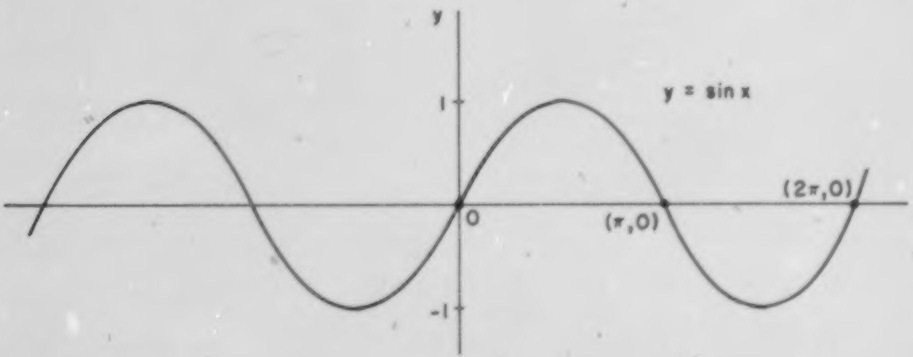


Figure 6-1

We have chosen the usual rectangular coordinate system, using x and y as abscissa and ordinate respectively, and obtained the familiar and beautiful sine curve. Do you see the relation between the shape of this curve and the related words: sinuous, and insinuate?

We could have chosen a polar coordinate system for a graphic representation of $y = \sin x$. We may expect a different looking graph on a different grid, but we should expect also to have some geometric counterparts of the algebraic properties we mentioned earlier.

When we use polar coordinates we customarily use as variables not x and y but r and θ . r is now a measure of the polar distance to the point (r,θ) , and θ is a measure of the angle between the polar axis and the polar ray through (r,θ) . In this context some authors say that r is a measure of the distance or modulus, and that θ is a measure of the argument, or amplitude.

A strong note of caution must be made in discussions of polar graphs of equations. From the fact that a point does not have a unique representation in polar coordinates we expect that a set of points may have several, perhaps quite dissimilar analytic representations. Any discussion of the relation between a graph and its analytic representation in polar coordinates must take account of this lack of uniqueness. We remember that a point P is on the graph of $r = f(\theta)$ if P has at least one pair of polar coordinates which satisfy this equation. Thus the point $P = (10,5)$ is on the polar graph of $r = 2\theta$, because $10 = 2(5)$, but the same point could also have been located by the coordinates $(10,5 + 2\pi)$, or $(-10,5 + \pi)$, or others, where the coordinates do not satisfy the equation $r = 2\theta$.

The polar graph of $r = \sin \theta$ is given in Figure 6-2. Can you now interpret the graph to show that r is a periodic bounded function of θ? We may note that the related polar equation for this graph is $r = - \sin (\theta + \pi) = \sin \theta$, hence is identical with the original polar equation.

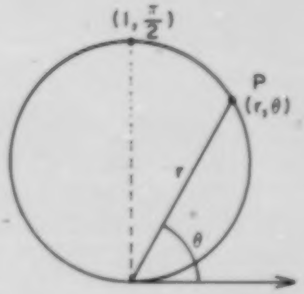


Figure 6-2

Both Figure 6-1 and 6-2, which are graphic representations of $y = \sin x$ exhibit a geometric property called symmetry. The algebraic counterpart of this property will be discussed in detail after the following exercises.

Exercises 6-2(a)

1. Give bounds for the graphs of the following equations.

   (a) $y = 2 \sin x$

   (b) $y = \sin 2x$

   (c) $y = 2 - \sin 2x$

   (d) $y = \frac{1}{2} \sin 2x$

   (e) $y = 4 + 2 \sin (3x + \frac{\pi}{2})$

   (f) $y = 0.6 \sin x + 0.8 \cos x$

   (g) $y = 2 \sin x + 3 \cos x$

   (h) $y = a \sin x + b \cos x$

   (i) $y = \sin^2 x$

   (j) $y = \sin^2 x - \cos^2 x$

2. Express in terms of a, b, c, d the bounds and the period of the graph of $y = a + b \sin (cx + d)$.

6-2(b) *Symmetry*

The graph in Figure 6-1 is symmetric with respect to the origin (and many other points), and to the line $x = \frac{\pi}{2}$ (and many other lines). The graph in Figure 6-2 is symmetric with respect to the point $(\frac{1}{2}, \frac{\pi}{2})$, and to the line $\theta = \frac{\pi}{2}$ (and many other lines). We shall concern ourselves with only the types of symmetry you have already met in earlier courses. We give their definitions here for the sake of completeness.

Point Symmetry. Given a set of points S , and a fixed point M . S is symmetric with respect to M if, for each point P of S there is a corresponding point P' of S such that M is the midpoint of $\overline{PP'}$. (The point P' is called the point-symmetric image of P with respect to M , or, when the context makes the reference clear, the image of P with respect to M.)

Line Symmetry. Given a set of points S , and a fixed line L . S is symmetric with respect to L if, for each point P of S there is a corresponding point P' of S such that L is the perpendicular bisector of $\overline{PP'}$. L is sometimes called an *axis* of *symmetry* of the set S , which may have more than one such axis. We sometimes borrow terminology from the applications, and call L an *axis* of *reflection*; in that case we may also call P' the *reflected image* of P with respect to L , or simply, the *reflection* of P in L .

In rectangular coordinates we readily establish an algebraic test for symmetry with respect to the origin. The point $P_1 = (x_1,y_1)$ has the image $P'_1 = (-x_1,-y_1)$ with respect to the origin. If P_1 is on the graph of $f(x,y) = 0$ then $f(x_1,y_1) = 0$. If the graph is symmetric with respect to the origin, for each point $P_1 = (x_1,y_1)$ on it, the graph must also contain the point $P'_1 = (-x_1,-y_1)$. That is, whenever $f(x_1,y_1) = 0$ we must also have $f(-x_1,-y_1) = 0$. This yields our test:

> The graph of an equation in rectangular coordinates is symmetric with respect to the origin if an equivalent equation is obtained by replacing (x,y) by $(-x,-y)$.

We may now test the equation $y = \sin x$, which may be written $y - \sin x = 0$. If we designate the left member as $f(x,y)$, we have: $f(-x,-y) = -y - \sin(-x)$, or $-y + \sin x$, or $-(y - \sin x)$, or $-f(x,y)$. This is clearly equal to zero whenever $f(x,y)$ is equal to zero; therefore, the graph is symmetric with respect to the origin.

As a second example we may test the equation $y = x^3$ whose graph is called a cubic parabola. If we write this equation as $y - x^3 = 0$ and call the left member $f(x,y)$, then we find $f(-x,-y) = (-y) - (-x)^3 = -y + x^3 = -(y - x^3) = -f(x,y)$. Clearly this is zero whenever $f(x,y) = 0$, thus our test for symmetry is satisfied, and the graph is symmetric with respect to the origin.

The test for symmetry with respect to any point $M = (h,k)$ other than the origin, is not at all difficult, but will not be presented here. If a curve has such symmetry, we can usually find a simpler analytic representation for it if we use the center of symmetry as a new origin.

In rectangular coordinates we can find a simple algebraic test for symmetry with respect to the axes. The point $P = (x,y)$ has the image $P' = (-x,y)$ with respect to the y-axis, and $P'' = (x,-y)$ with respect to the x-axis.

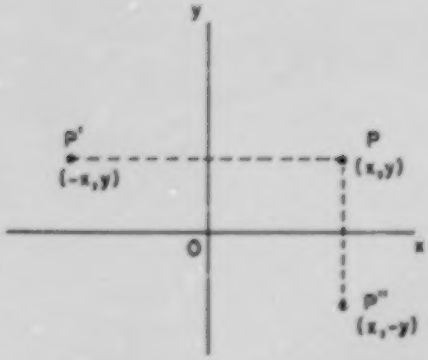


Figure 6-3

These relations lead to our test. If the graph is symmetric with respect to the y-axis, then, for each point $P_1 = (x_1,y_1)$ on the graph there must be a point $P_1' = (-x_1,y_1)$ also on the graph; that is, if $f(x_1,y_1) = 0$, so also must $f(-x_1,y_1) = 0$. This means that the equations $f(x,y) = 0$, and $f(-x,y) = 0$ must be equivalent equations. We show that the graph of $y = \sin x$ in rectangular coorinates does not have this type of symmetry. This equation can be written as $y - \sin x = 0$, or $f(x,y) = 0$. Then $f(-x,y)$ is $y - \sin(-x)$ or $y + \sin x$, which clearly need not equal zero when $f(x,y) = y - \sin x$ does.

The test for symmetry with respect to the x-axis is analogous and we summarize these two tests:

> The graph of an equation in rectangular coordinates is symmetric with respect to the
>
> (a) x-axis, if an equivalent equation is obtained by replacing (x,y) by $(x,-y)$;
>
> (b) y-axis, if an equivalent equation is obtained by replacing (x,y) by $(-x,y)$.

It is quite possible for a graph to be symmetric with respect to both axes. The graph of $x^2 + 4y^2 = 36$ is an ellipse and it exhibits such double symmetry both algebraically and geometrically.

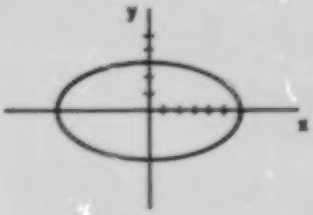


Figure 6-4

If y can be expressed as an explicit function of x, $y = f(x)$, such that $f(x)$ contains only even powers of x then we say that y is an even function of x, and recognize that its graph is symmetric with respect to the y-axis. Some examples of even functions of x are:

$$y = 7x^2 \, , \; y = x^2 + 3x^4 \, , \; y = \sqrt{x^4 - 3x^2} \, , \; y = 2x^2 - \frac{1}{x^6} \, .$$

Note that the equation $x^2 + 4y^2 = 36$ does not define y as a function of x, or x as a function of y. Rather, it yields expressions for y as two (even) functions of x, that is, $y = \frac{1}{2}\sqrt{36 - x^2}$, and $y = -\frac{1}{2}\sqrt{36 - x^2}$. The graphs of these functions are semi-circular arcs each of which is, in fact, symmetric with respect to the x-axis.

Where x and y are related implicitly by an equation $f(x,y) = 0$, we may still use the concepts above. If $f(x,y)$ contains only even powers of x, then $f(x,y) = f(-x,y)$, and the graph of $f(x,y) = 0$ will be symmetric with respect to the y-axis. Thus we may still relate the symmetry of the graph to even functions even when those functions are implicit. Some examples of even implicit functions are:

(a) $x^2y + x^4y^2 = 10$, whose graph is symmetric with respect to the y-axis but not the x-axis;

(b) $x^2y^2 + 3xy^4 + 2x = 0$, whose graph is symmetric with respect to the x-axis but not the y-axis;

(c) $x^2y^4 + 2x^2 + 3y^2 = 4$, whose graph is symmetric with respect to both axes.

Note that the graph of $x^2 + 4y^2 = 36$ is symmetric with respect to the origin also, since $f(x,y) = f(-x,-y)$. Which, if any, of the graphs of a, b, and c, above, are symmetric with respect to the origin?

Symmetry with respect to other lines will not be generally discussed here, but there is a simple test for symmetry with respect to the lines which bisect the angles formed by the axes. These lines are L_1: $y = x$, and L_2: $y = -x$. The reflection of $P = (x,y)$ in L_1 is $P' = (y,x)$, and in L_2 is $P' = (-y,-x)$ as may be seen in the figure.

The corresponding test follows as before and may be stated thus:

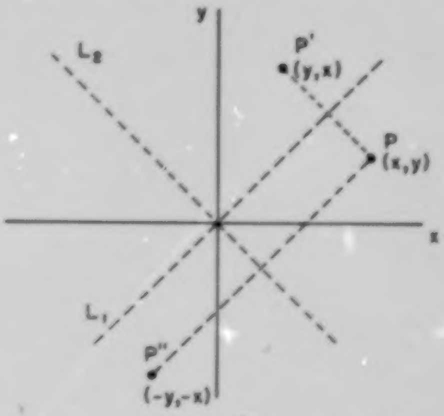


Figure 6-4

The graph of an equation in rectangular coordinates is symmetric with respect to the line,

(a) $y = x$, if an equivalent equation is obtained by replacing (x,y) by (y,x);

(b) $y = -x$, if an equivalent equation is obtained by replacing (x,y) by $(-y,-x)$.

Examples:

1. The graphs of the following equations are symmetric with respect to the line $y = x$;

(a) $xy = 6$

(b) $xy = x^3 + y^3$

(c) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

(d) $x + y = 10$

(e) $x^2 + y^2 - 6x - 6y = 12$.

2. The graphs of the following equations are symmetric with respect to the line $y = -x$;

(a) $xy = 6$

(b) $y = x + 3$

(c) $x^2 + y^2 - 6x + 6y = 12$

(d) $x^3 = y^3 + xy$

(e) $y = x^2y^2 + x$.

If a graph has an axis of symmetry parallel to the x-axis or the y-axis it may have a simpler analytic representation if we use new coordinates based on this axis of symmetry. Such transformations of coordinates are considered in detail in Chapter 10. Tests for symmetry with respect to other lines than those mentioned are available, but they are beyond the scope of this book.

These comments on symmetry in rectangular coordinates have their counterparts in polar coordinates. Point symmetry with respect to the pole requires that the graph of $f(r,\theta) = 0$ contain, for each point $P = (r_1,\theta_1)$ the corresponding point $P' = (-r_1,\theta_1)$. This condition will be satisfied if $f(r,\theta)$ is an even function of r. Note that the condition is sufficient to establish such symmetry but it is not necessary. Thus, the graph of $r = 5$ is a circle with radius 5, and it does have such symmetry, but this equation does not define an even function of r. We will not analyze the general situation, but note that $r = 5$ and $r = -5$ are related polar equations for the same circle. These equations may be written as $r - 5 = 0$ and $r + 5 = 0$, and then combined as in Chapter 5 by multiplying corresponding members to get $r^2 - 25 = 0$. This equation does give an even function of r and its graph, which is the same as that of $r = 5$ and of $r = -5$, is therefore symmetric with respect to the pole.

The point $P = (r,\theta)$ has, as its image with respect to the line containing the polar axis, the point $P' = (r,-\theta)$. We will not treat line symmetry in general, but we note an easy test for symmetry with respect to any line through the pole, say the line $\theta = k$. In this case the points $P = (r,k + \alpha)$ and $P' = (r,k - \alpha)$ are line-symmetric images for any value of α.

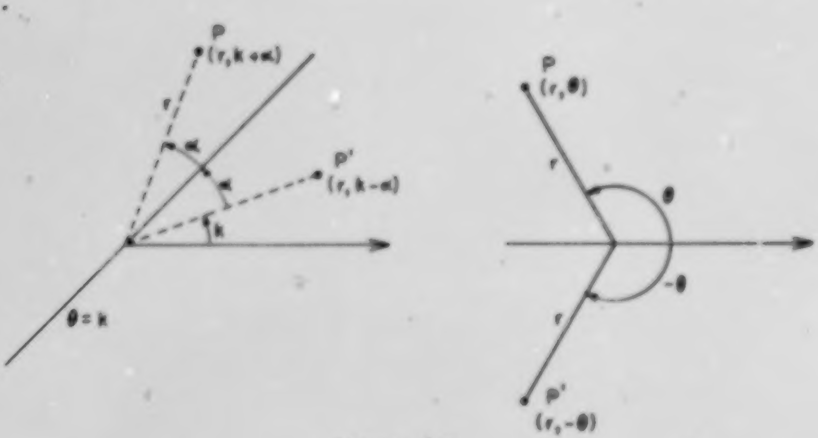


Figure 6-5

We state a test for such symmetry:

The polar graph of an equation is symmetric with respect to the line $\theta = k$ if an equivalent equation is obtained by replacing $(r,k + \alpha)$ by $(r,k - \alpha)$. In particular, the graph will be symmetric with respect to the line along the polar axis if $f(r,\theta) = f(r,-\theta)$.

These should again be recognized as sufficient but not necessary conditions. Since we have infinitely many polar representations of the symmetric points P and P' we could have infinitely many test for such symmetry. The test we have presented is the simplest to apply, and, with the concept of related polar equations, is adequate for the work of this course.

If we go back to an equation from Example 1, $r = \sin \theta$, we may write it $r - \sin \theta = 0$, and call the left member of this equation $f(r,\theta)$. The diagram there suggests that the line $\theta = \frac{\pi}{2}$ is an axis of symmetry and we compare $f(r,\frac{\pi}{2} + \alpha)$ and $f(r,\frac{\pi}{2} - \alpha)$. The first of these becomes $r - \sin(\frac{\pi}{2} + \alpha)$, or $r - \cos \alpha$. The second of these becomes $r - \sin(\frac{\pi}{2} - \alpha)$ or $r - \cos \alpha$. The identity of these expressions established the line symmetry of the graph, as indicated. We may have stated, in corresponding manner, that the point $P = (r,\frac{\pi}{2} + \alpha)$ is on the curve if and only if the corresponding point $P' = (r,\frac{\pi}{2} - \alpha)$ is on the curve. This is, in effect, what we have shown.

Exercises 6-2(b)

1. May a set of points have two centers of symmetry? Discuss your answer, with examples.
2. Give an example of a set of points which has exactly 2 axes of symmetry; exactly 3 ; exactly 4 .
3. Give an example of a set of points which has an infinte number of axes of symmetry.
4. If a graph is symmetric with respect to both axes must it be symmetric with respect to the origin? Illustrate.
5. If a graph is symmetric with respect to the origin must it be symmetric with respect to both axes?

6. Discuss the symmetry of the graphs of each of the equations listed:

(a) $x^2 + y^3 = 16$

(b) $x^3 - y^3 = x + y$

(c) $y = x^2 - 2x^4 + 5x^6$

(d) $x(x^2 + y^2) = y(x^3 + y^3)$

(e) $x^2y + xy^2 = 1$

(f) $(x + y)^2 + 2(x + y) = 1$

(g) $(x + y)^2 + 3(x + y) = 1$

(h) $x^2 + y^3 = y^2 + x^3$

(i) $x^4 + x^2y^2 + y^4 = x + y^2$

(j) $x^n + y^n = 1$

(k) $r^2 = \sin\theta$

(l) $r = \sin^2\theta$

(m) $r = 2 + \sin(\theta + \pi)$

(n) $r = \dfrac{6}{1 + \cos\theta}$

(o) $r = \dfrac{6}{3 - \cos(\theta - \frac{\pi}{2})}$

(p) $r^2 \cos^2\theta = 10$

(q) $r^2 = \sin 2\theta$

(r) $r = 2 \sin 3\theta$

(s) $r = 3 + 2\cos(\theta + \frac{\pi}{2})$

(t) $r = a + b \sin\theta$

Challenge Problems

1. (For discussion) By analogy with line symmetry in two dimensions, consider symmetry with respect to a plane in three dimensions. We are familiar with our reflected images in a mirror and accept the fact that there is a "reversal" of some sort. The reflection of my right hand is the "left hand" of my reflected image. Why is this reversal only left-right? Why is there not also a reversal of top-bottom, so that my reflected image would appear to stand on its head?

2. Given the line L: $ax + by + c = 0$ and the point $P_1 = (x_1, y_1)$ not on the line. Find coordinates for $P_2 = (x_2, y_2)$, the symmetric image of P_1, with respect to L.

6-2(c) Extent.

We discussed the equation $x^2 + 4y^2 = 36$ earlier from the point of view of symmetry. We use it now to discuss the extent of a graph. This equation yields two equations which define y as a function of x, $y = \frac{1}{2}\sqrt{36 - x^2}$ and $y = -\frac{1}{2}\sqrt{36 - x^2}$. We see that if we take values of

$|x|$ large enough we shall have in both cases corresponding values of y which are imaginary. Since our graphs consider only real values of x and y we now inquire about possible values of x which will lead to real values of y, and vice versa. In these cases we must have $-6 \leq x \leq 6$, or $|x| \leq 6$. For these restricted values of x the corresponding values of y range from -3 to 3. The geometric versions of these restrictions can be applied to the graphs of both functions of x defined above, but it is more useful to consider the union of these graphs; that is, the graph of the original equation

$$x^2 + 4y^2 = 36 .$$

From the discussion above we see that the points of the graph all lie in a rectangular region 12 units wide and 6 units high, centered at the origin. If, in general, we can express y as a function of x, and there are such restrictions on values of x as will yield only real values for y, we say that the domain of the function is bounded. Thus, all points of the graph of the function $y = \frac{1}{2}\sqrt{36 - x^2}$ are confined to a strip bounded by two vertical lines, $x = \pm 6$, as indicated in Figure 6-6. If, in general, the possible real values of y are similarly restricted, we say that the range of the function is bounded. Thus, all points of the graph of $y = \frac{1}{2}\sqrt{36 - x^2}$ are confined to a strip bounded by two horizontal lines, $y = \pm 3$, as indicated in Figure 6-6. If both the domain and range of a function are bounded, we say that the function is bounded, in which case its graph is confined to the intersection of a vertical and horizontal strip, and is therefore confined to a rectangular region. These terms are usually applied to equations and their graphs even when the functions are only defined implicitly. Thus, when we say that the graph of $x^2 + 4y^2 = 36$ is bounded, we indicate that it is contained in a rectangle, as mentioned earlier.

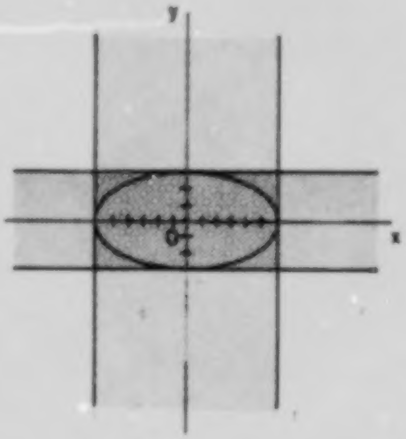


Figure 6-6

If the equation were $x^2 - 4y^2 = 36$, we would obtain

$$y = f(x) = \pm \frac{1}{2}\sqrt{x^2 - 36}\ .$$

We now note that we must take values of $|x|$ large enough to make the radicand non-negative; that is, $|x| \geq 6$, which will be true if either $x \geq 6$ or $x \leq -6$. Geometrically, this means that y is defined only for points on the edges or outside the vertical strip bounded by the lines which are the graphs of $x = 6$ and $x = -6$. With these restrictions on x we may now have any value of y. The original equation yields two equations which define x as a function of y, $x = \sqrt{36 + 4y^2}$ and $x = -\sqrt{36 + 4y^2}$ and we see that x in both cases is defined for all values of y. It is not customary, in this case, to speak of $y = \frac{1}{2}\sqrt{x^2 - 36}$ as a bounded function, but merely to say that the domain of x excludes certain values.

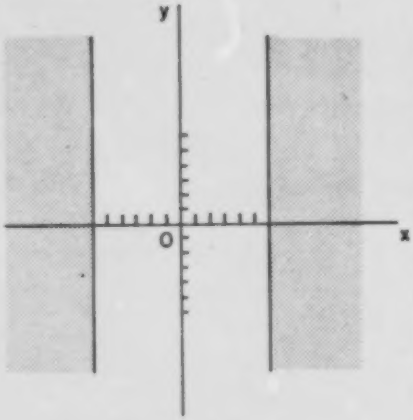


Figure 6-7

Another concept emerges when we consider $y = \frac{1}{x}$. The domain of x is also restricted here since x cannot equal zero. With this exception, y is defined for all values of x. Geometrically, points of the graph are available except at the places where the abscissa is zero, therefore this graph does not touch or cross the y-axis. If we write the equation $x = \frac{1}{y}$, we see that the graph does not touch or cross the x-axis. Also, from the fact that $xy = 1$, we must have x and y either both positive or both negative, which means, geometrically, that we are confined to the first and third quadrants exclusively. From the equation $xy = 1$ we see also that as we take points of the graph nearer the x-axis we must take them farther from the y-axis, and vice-versa. A line, such as the x-axis in this case, to which points of the graph approach more and more closely, but which contains no point of the graph, is called an asymptote of the graph. The graph of $y = \frac{1}{x}$ has two asymptotes; namely, the x-axis and the y-axis. Our examples will illustrate the treatment of asymptotes in several situations, but we make a general observation. If our analytic representation can be written as

$$y = \frac{f(x)}{g(x)},$$

where $g(x)$ may equal zero for some value of x, say $x = a$ then, for this value of x, y is not defined. Also, if, $f(b) \neq 0$ then, in general, as we take values of x closer to b the corresponding values of y become greater in absolute value. Geometrically this usually means that as we take points closer to the line $x = b$ they must be farther from the x-axis. Thus, the line $x = b$ is a vertical asymptote. If $g(x) = 0$ has roots b_1, b_2, ..., and these are not roots of $f(x) = 0$, there will, in general, be vertical asymptotes, $x = b_1$, $x = b_2$, ... There is no difficulty in revising these comments to apply to horizontal asymptotes: If we can write $x = \frac{h(y)}{k(y)}$; and $k(y) = 0$ has roots c_1, c_2, ..., and these are not roots of $h(y) = 0$, then, in general, there will be horizontal asymptotes, $y = c_1$, $y = c_2$, ...

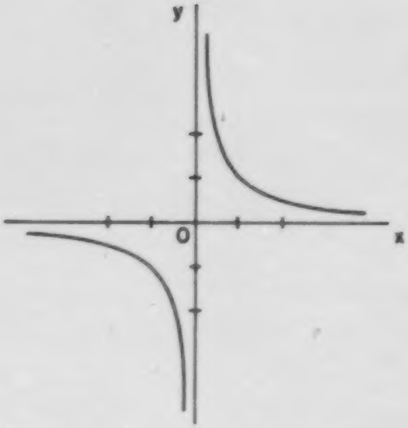


Figure 6-8

Example: Discuss and sketch the graph of

$$y = \frac{x}{x^2 + 2x - 3}.$$

Solution: The equation can be written as $y = \frac{x}{(x + 3)(x - 1)}$; hence, from the discussion above, the curve has as vertical asymptotes the lines $x = -3$ and $x = 1$. y is not defined for these values of x, but y is defined for all other values of x. If $x > 1$ and increasing then

y is positive, and decreasing. For large values of x the values of $x + 3$ and $x - 1$ are relatively close to values of x, and y is relatively close to $\frac{1}{x}$, which is positive; therefore, the corresponding points of the curve are close to the x-axis. If $0 < x < 1$ the numerator is positive and the denominator negative; therefore, y is negative. The curve still approaches the line $x = 1$ as an asymptote, but from the other side. If $-3 < x < 0$ the numerator and denominator are both negative, therefore y is positive. As before, the curve approaches the line $x = -3$ as an asymptote. If $x < -3$ then the numerator is negative, the denominator positive, and y negative. The curve again approaches the line $x = -3$ as an asymptote, but from the left side.

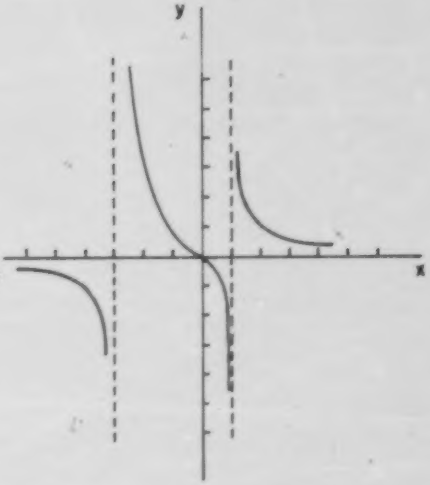


Figure 6-9

For negative values of x with large absolute value the values of $x + 3$ and $x - 1$ are relatively close to x, and the corresponding value of y is relatively close to $\frac{1}{x}$, which is now negative. That is, as we take points of the graph farther to the left, they must be closer to the x-axis, from below. The graph, pictured in Figure 6-9, shows that algebraic and geometric relationships we have discussed.

A discussion of the appearance of a graph for large values of $|x|$ or $|y|$, whether we take x and y positive or negative, is part of the discussion of the extent of the graph, and is sometimes referred to as a discussion of the behaviour of the graph for extreme values of the variables.

The concept of excluded values because of a zero denominator has one further application. Consider

$$y = x + 2 \text{ , and } y = \frac{x^2 - 4}{x - 2} .$$

It would not be correct to write the second equation as

$$y = \frac{(x + 2)(x - 2)}{(x - 2)}$$

and then remove the common factor

$$y = \frac{(x + 2)\cancel{(x - 2)}}{\cancel{(x - 2)}}$$

to arrive at the first equation

$$y = x + 2 .$$

As a matter of fact, the two equations and their graphs are different in a small but significant way. In the first equation, y is defined for all x ; in the second equation y is defined for all x , except $x = 2$. Geometrically, the graph of the first equation is a line; the graph of the second equation is a line except for a missing point at the place where $x = 2$, that is, it is an interrupted line. (Could you interrupt this line at the place where $x = 1$, also?)

The discussion of these excluded points, lines, or regions is useful in describing the extent of the graph. It's all very well to know where the graph does not go, but we are still concerned with the points through which it does go, that is, with drawing the graph. The most straightforward way of drawing the graph of an equation is to plot a number of points on it and draw a curve through them. If the equation has the form $y = f(x)$ you can make a table showing the value of y corresponding to each of a number of values of x . You have done this many times in the past, and there is no need to go into detail again here. However, it is worth reminding you that you should think about how many values of x to use, and which ones, and how to join the corresponding points.

As in an election poll, we take enough samples, with special attention to certain critical spots, until we have some reasonably clear idea of how the whole picture will look. There will always be some disagreement about how many are "enough", and what is "reasonably clear". Our sampling can start at some easily available points. On our grid we can most easily find the places where the graph crosses the axes. Since the x-axis, for example, has the equation $y = 0$, we may solve simultaneously: $y = 0$, $y = f(x)$; that is, we may find the roots of the equation $f(x) = 0$, in order to find the abscissas of these crossing points. If $f(x) = 0$ has roots a_1 , a_2 , ... , then these numbers are the x-intercepts of the graph, which goes through the points $(a_1,0)$, $(a_2,0)$, ... These points are easily plotted on the grid, as are the points of intersection of the graph with the y-axis. But, no matter how many points you plot, there always remains the question of how the curve behaves elsewhere. It is to cast further light on this question that you should investigate, before any extensive computation, the properties of the curve and its analytic representation in the manner we have just illustrated. We summarize this type of investigation in mnemonic form: "Check the SEPIA first." (Symmetry, Extent, Periodicity, Intercepts, Asymptotes.)

The curves and equations with which we deal in this course are reasonably well behaved, and the points of the graph are usually smoothly connected, with certain notable exceptions. We have already dealt with graphs of inequalities in Chapter 5 , and will not deal with them at great length here, but will consider them in the examples whenever there is any matter of special interest.

A curve usually separates the plane locally into two regions (above and below, inside and outside, ...). In many cases in this text the points in these two regions are precisely those whose coordinates satisfy one or the other of the inequalities we obtain from the original equation. Thus the graph of $x^2 + y^2 = 25$ is a circle of radius 5 , centered at the origin. The graph of $x^2 + y^2 < 25$ is the interior of that circle, and the graph of $x^2 + y^2 > 25$ is the exterior.

We have used rectangular coordinates in this general discussion, but much of it can be adapted to polar coordinates, though the graphs will not have the same geometric properties. In polar coordinates the graphs of inequalities are sometimes unexpected. Thus the graph of $r = 5$ is a circle, the graph of $r > 5$ is the region outside that circle, but the graph of $r < 5$ is the entire plane. The graph of $r = \frac{1}{\theta}$ is only a remote cousin to the graph of $y = \frac{1}{x}$. The rectangular graph (a hyperbola) has a vertical asymptote, the line $x = 0$, and this is a geometric consequence of the fact that y is not defined for $x = 0$. From the equation $r = \frac{1}{\theta}$, we see that r is not defined for $\theta = 0$; nevertheless the line $\theta = 0$ contains the point $P = (-\frac{1}{\pi}, 0)$. This point has infinitely many other polar representations, including particularly $P = (\frac{1}{\pi}, \pi)$, and since these coordinates satisfy the equation $r = \frac{1}{\theta}$, we must allow P on the graph of $\theta = 0$. There are, as a matter of fact, infinitely many other points for which we can find some pair of polar coordinates that satisfy $r = \frac{1}{\theta}$, and which lie on the line $\theta = 0$. Therefore this line is not an asymptote for the graph if $x = \frac{1}{\theta}$.

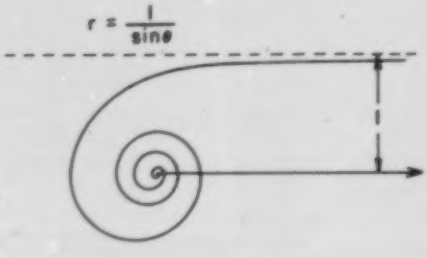


Figure 6-10

The graph of $r = \frac{1}{\theta}$ does, nevertheless, have a true asymptote, the line corresponding to $r = \frac{1}{\sin\theta}$, but the discussion of this must consider the value of $\frac{\sin\theta}{\theta}$ as θ gets closer to 0 , and this discussion is beyond the scope of this book.

We will, in the examples and text that follow, use polar representation or any other that seems appropriate to the problem and our purposes, and carry the discussion to the level and detail that seem fitting. Our examples will illustrate the general principles above, and some ideas of less general application, but the student is urged to extend his own experience by doing as many of the exercises as he can. One suggestion we have found valuable: an equation and its graph should be considered in a dynamic, rather than a static way. If we have $y = f(x)$, what happens to y when x increases a little, when x approaches 0 , when x gets very large? If we have a point $P_0 = (x_0, y_0)$ of the graph, how does the curve look, near that point? Think of the point as moving along the curve, and our analysis as a moving picture of the point rather than a snapshot of the entire curve.

6-3. Conditions and Graphs (Rectangular Coordinates)

In this section we shall discuss a number of examples in detail. This discussion will bring together and apply a number of topics you first studied separately. We shall illustrate also some useful approaches that may be new to you.

Example 1. Discuss and sketch the graph of $y = x + \frac{1}{x}$.

Solution. There is no symmetry with respect to either axis, since we do not get equivalent equations by replacing x by $-x$; or y by $-y$. There is symmetry with respect to the origin, because we do get an equivalent equation by replacing x by $-x$ and y by $-y$. There is a vertical asymptote, the y axis, whose equation is $x = 0$. For large $|x|$ and x either positive or negative, y and x become relatively equal, since $\frac{1}{x}$ becomes relatively small. Geometrically this means that the graph approaches the line $y = x$ asymptotically, from above, on the right, and from below, on the left.

We shall graph this equation in a way which may be new to you, by addition of ordinates. You can draw fairly accurate graphs of $y = x$ and $y = \frac{1}{x}$ with almost no effort. Do so, with respect to the same axes. Then, for each of a number of different values of x , add the y-coordinates of the points on the two curves with that x-coordinate. The result is the y-coordinate of the corresponding point on the graph of $y = x + \frac{1}{x}$. The addition can be done using marks on the edge of a piece of paper, but you must pay attention to the algebraic signs. The sketch below illustrates the process.

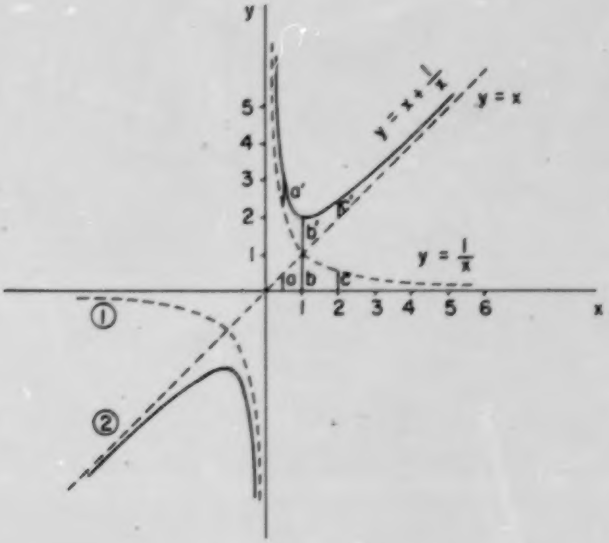


Figure 6-11

We suggest this sequence of steps:

1. Draw the familiar curves ① and ②.
2. At several points along the x-axis erect perpendiculars to meet the two curves. In Figure 6-11 the ordinate segments, a , b , c , were found this way at $x = \frac{1}{2}$, $x = 1$, $x = 2$. (We shall refer to these ordinate segments simply as the ordinates.)
3. Add the corresponding ordinates for the two curves with due regard to sign. In Figure 6-11, a , the ordinate at $x = \frac{1}{2}$ is raised to a' above the hyperbola; b is raised to b' above the hyperbola; c is raised to c' above the line; and so on.
4. Connect the new points thus found, to get the new curve.

Example 2(a) Sketch the graph of $y = x^2 + 2$.

Example 2(b) Sketch the graph of $y = \sin x - 3$.

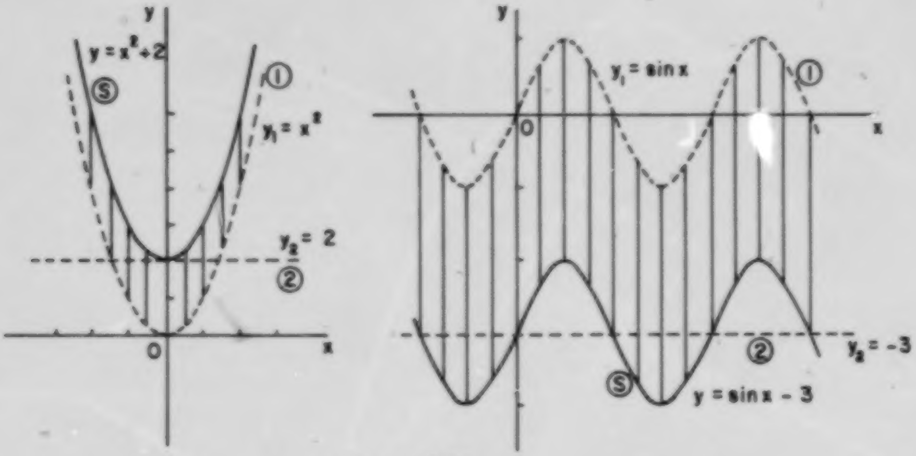


Figure 6-12

Solution 2(a). Draw the familiar graphs of $y_1 = x^2$, indicated by ① in the figure and of $y_2 = 2$, indicated by ② in the figure. Then "raise" every point of ① 2 units, as indicated by the dashed lines, to get the graph Ⓢ of $y = y_1 + y_2 = x^2 + 2$.

2(b) The solution should be clear from the figure and is left to the student.

The process of graphing by subtraction of ordinates is related to the process of graphing $y = -f(x)$ from the graph of $y = f(x)$. The discussion of symmetry in the previous section indicates immediately that these two graphs are symmetric images of each other with respect to the x-axis. That is, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$, with respect to the x-axis.

Example 3(a). Sketch the graph of $y = - x^2$.

Example 3(b). Sketch the graph of $y = - \cos x$.

Solution: (Refer to Figure 6-13)

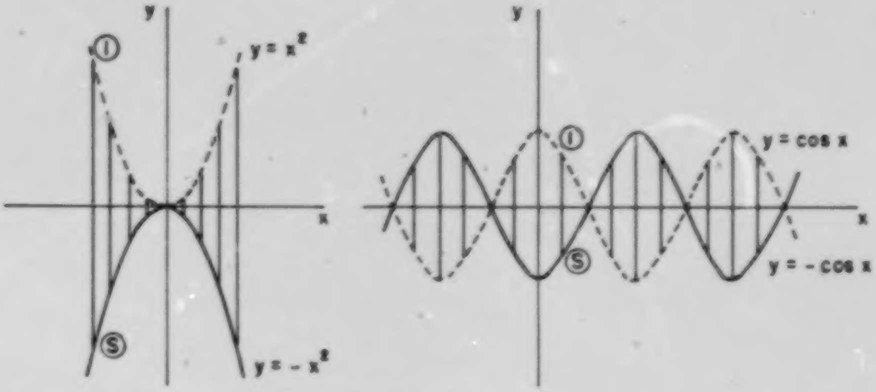


Figure 6-13

3(a) Construct the familiar graph ① of $y = x^2$; then extend the ordinate of each point of ① down its own length through the x-axis to get the reflected points, which we connect to obtain the solution, Ⓢ.

3(b) The solution, indicated in Figure 6-13, is left to the class.

We may now sketch graphs by subtracting ordinates, since, if y $y = f(x) - g(x)$, then $y = f(x) + (-g(x))$.

Example 4(a). Sketch the graph of $y = 3 - x^2$.

Example 4(b). Sketch the graph of $y = 1 - \sin x$.

Solution 4(a). (Refer to Figure 6-14).

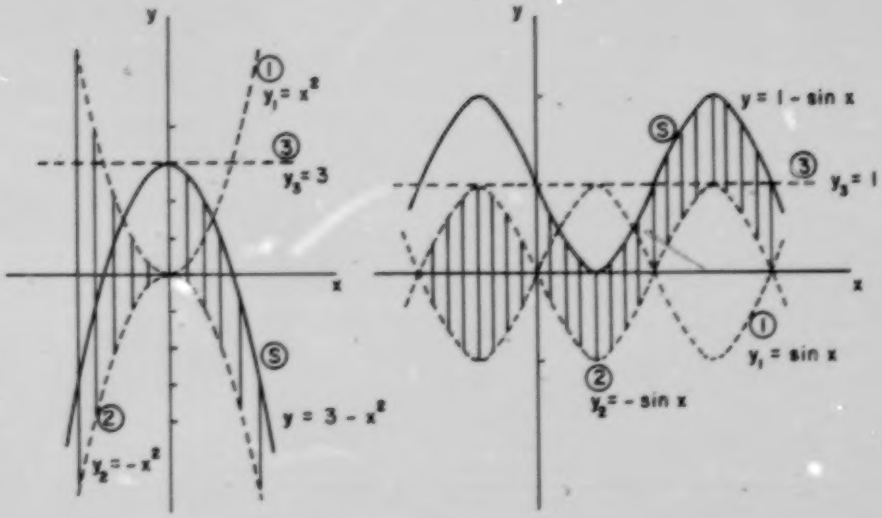


Figure 6-14

We suggest these steps:

(1) Draw the familiar graphs ①: $y_1 = x^2$, and ③ $y_3 = 3$.

(2) Reflect ① with respect to the x-axis to get ②: $y_2 = -x^2$.

(3) Add the ordinates for ② and ③ to get Ⓢ: $y = 3 - x^2$.
This last step is equivalent to adding 3 units to each ordinate of ②, as indicated on the graph.

We may extend these graphical methods to the multiplication of ordinates. We have already done this in some cases but not with this terminology. The graph of $y = 2 \sin x$ illustrates a simple application of this method. We compare this graph with the graph of $y_1 = \sin x$ and recognize that when $y_1 = 0$ then $y = 0$; when $y_1 > 0$ then $y > 0$; and when $y_1 < 0$, then $y < 0$. We just draw the graph of $y_1 = \sin x$, and double the ordinates to find corresponding ordinates for $y = 2 \sin x$. It is as if the graph were stretched, vertically, away from the x-axis.

Example 5(a). Sketch the graph of $y = 2 \sin x$.

Example 5(b). Sketch the graph of $y = 2x^2 - 8$.

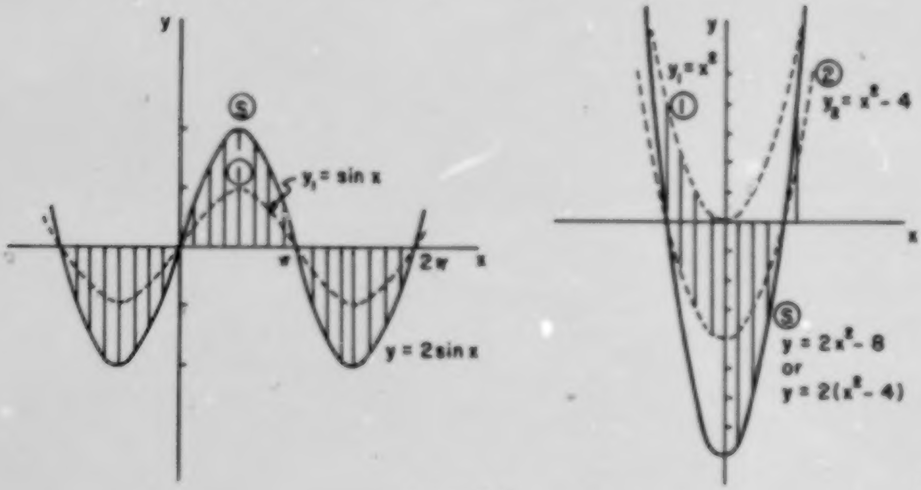


Figure 6-15

Solution 5(a). We sketch the familiar graph, ①: $y_1 = \sin x$, then double each ordinate of ① to get the graph, Ⓢ: $y = 2 \sin x$. Note that for $0 < x < \pi$ we have $0 < y_1 < 1$, therefore $0 < 2y_1 < 2$. Thus Ⓢ is bounded between 2 and -2. If, more generally, $y = a \sin x$, then y is bounded between $|a|$ and $-|a|$. In this case $|a|$ is called the **amplitude** of this sine curve. It is the measure of the maximum departure of points of the curve from the x-axis, and has important physical applications.

Solution 5(b). We have illustrated the sequence of graphs: ①: $y_1 = x^2$; ②: $y_2 = x^2 - 4$; ③: $y = 2(x^2 - 4)$. We could have found the same graph with the sequence ①: $y_1 = x^2$; ②: $y_3 = 2x^2$;

③: $y = 2x^2 - 4$. We leave the details to the student.

We may in general relate the graph of $y = bf(x)$ to that of $y_1 = f(x)$ if b is a constant. Both graphs cross the x-axis at the same points. If $b > 0$ then both graphs are above or below the x-axis together. If $b < 0$ then the graphs of $y = -bf(x)$ and $y_1 = f(x)$ are together above or below the x-axis. In this latter case we graph $y_2 = |b|\ f(x)$, then reflect this graph in the x-axis to get the graph of $y = bf(x)$.

Example 6(a). Sketch the graph of $y = -2x^2$.

Example 6(b). Sketch the graph of $-3 \sin x$.

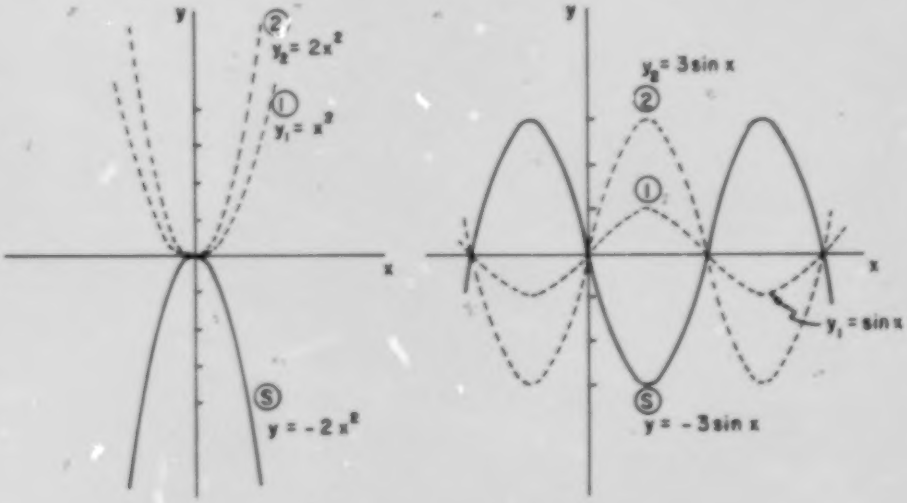


Figure 6-16

Solution 6(a). Sketch the familiar curve ①: $y_1 = x^2$. Double the ordinates, which in this case are all non-negative, to get ②: $y_2 = 2x^2$. Finally reflect ② in the x-axis to get ③: $y = -2x^2$.

Solution 6(b). We leave the solution to the student. Note that in Example 6(a) we could have used the sequence $y_1 = x^2$; $y_3 = -x^2$, $y = -2x^2$. That is we could have reflected, then stretched to get the final curve, in both 6(a) and 6(b). We leave these details to the student.

Our final cases concern multiplication of ordinates with variable factors. These are the most difficult, the most interesting, and the most useful of the applications of these methods of graphing by combinations of ordinates.

Example 7. Sketch the graph of $y = x^2 - x$.

Solution.

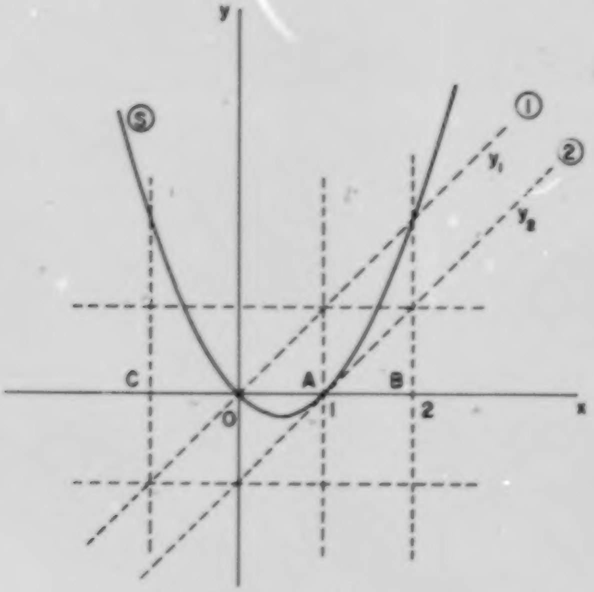


Figure 6-17

We could sketch the graph by subtraction of ordinates but we choose to illustrate the method of graphing by multiplication of ordinates. Thus $y = x(x - 1)$, and we draw the graphs ①: $y_1 = x$, and ②: $y_2 = x - 1$; two parallel lines. When $x < 0$ then y_1 and y_2 are both negative and their product, y , is positive. If $x < 0$ and decreasing then y is positive and increasing, and corresponding points of S are in the third quadrant.

Since $y = y_1 y_2$, clearly y must equal zero when either y_1 or y_2 equals zero, thus the graph S intersects the x-axis at A and B . Between O and A we have $0 < x < 1$, with ① above and ② below the x-axis. In this interval $y_1 > 0$, $y_2 < 0$ and therefore $y < 0$ and the graph is below the x-axis. Between A and B we have $1 < x < 2$ and both y_1 and y_2 positive, therefore $y > 0$. The graph indicates that since ① and ② are above the x-axis then Ⓢ must be also. However in that interval $0 < y_2 < 1$ therefore $y_2 y_1$ is a proper fractional part of y_1 , thus $y = y_2 y_1 < y_1$, therefore Ⓢ is above ② but below ①.

As x increases beyond B we have $x > 1$, y_1 and y_2 positive and increasing, and y increasing even more rapidly, thus Ⓢ is above both ① and ②.

We have taken this time to discuss the graph of what is, after all, only a parabola, because the analysis and method will help in more difficult and unfamiliar situations.

Example 8. Sketch the graph of $y = .1x \sin x$.

Solution. We are familiar with the graphs of $y_1 = .1x$, and $y_2 = \sin x$. Since sin x is a bounded periodic function of x we have $|y_2| \leq 1$ and $|y_1| \leq |.1x|$. The graph of this last condition is the pair of lines ① and ② in Figure 6-18.

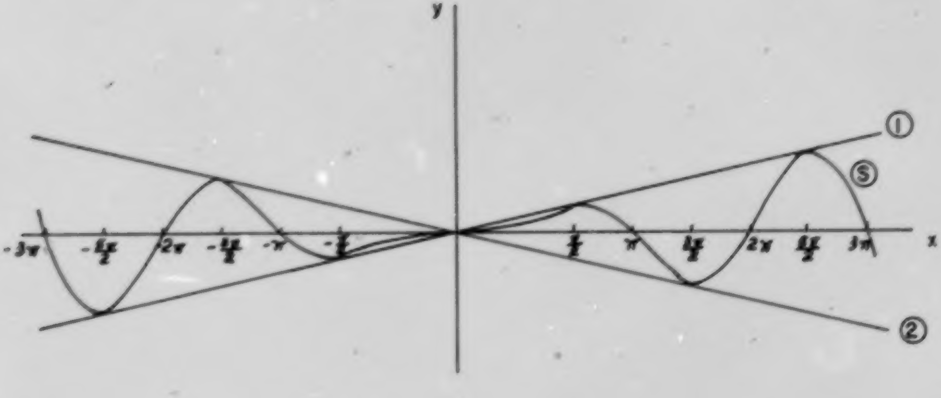


Figure 6-18

We have compressed the scale along the x-axis for the purpose of getting enough of the graph on the page to illustrate the discussion.

When $x > 0$, all points of the graph lie within, or on the boundary of the angular region formed by the right half-lines of ① and ②. Since $y = y_1y_2$, then y will equal zero when either y_1 or y_2 equals zero. y_1 is zero only at the origin, but y_2 is zero at integral multiples of π. Also, when $y_2 = 1$ we have $y = .1x$ and when $y_2 = -1$ we have $y = -.1x$, which means that the graph Ⓢ will touch alternately the lines 1 and 2 at points where $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

We leave the rest of the discussion of this graph to the student but mention an important application.

If we consider how the graph of $y_2 = \sin x$ is changed by the variable factor $y_1 = .1x$, we may think of the amplitude of y_2, as changed by this variable factor. In this example we may say that the amplitude of $\sin x$ is increasing linearly. If we had $y_3 = f(x) \sin x$ then we also have a sine wave whose amplitude is being changed or constrained by the variable factor $f(x)$. The graph of y_3 would be constrained by the symmetric curves: $y = f(x)$ and $y = -f(x)$ and would oscillate between them, touching them alternately when $x = \pi, 3\pi, 5\pi, \ldots$, as before.

This systematic changing of the amplitude is called amplitude modulation and is the basis for AM radio reception. A typical equation here would be $y = \sin 1000\pi t \sin 1000000\pi t$.

This graph would show a rapidly oscillating curve (the carrier or radio frequency, or RF wave) modulated by a less rapidly oscillating curve (the signal, or audio frequency, or AF wave).

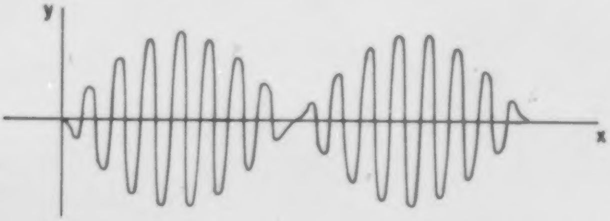


Figure 6-19
This sketch, not to scale, illustrates the idea.

The methods just discussed, for relating graphs of equations to graphs of more familiar equations by combining ordinates are called by some authors, composition of ordinates. We apply similar techniques in polar graphs in some examples later.

We consider now some further examples of graphs of equations in rectangular coordinates.

Example 9. $4x^2 - 9y^2 + 8x + 36y + 4 = 0$. From this equation it is not obvious whether the curve is symmetric with respect to any point or line, or whether it has any asymptotes. Nor can we easily see what parts of the plane it does or does not enter. We can find as many points on it as we have the patience for, since picking a value for x gives us a quadratic equation for y .

The sensible approach, however, is to use a trick you learned in algebra: complete the square in x and y . We get

$$4(x^2 + 2x + 1) - 9(y^2 - 4y + 4) = -4 + 4 - 36$$

or

$$\frac{(y-2)^2}{4} - \frac{(x+1)^2}{9} = 1 .$$

These numerators are related to distances from the lines $y = 2$ and $x = -1$, and we might expect a considerable simplification in the discussion of this graph if we had new coordinates based on these lines as axes. Such transformations are carried out more generally in Chapter 10, but we show the details here in order to continue with our discussion of the graph. If we let $u = x + 1$ and $v = y - 2$ the equation becomes

$$\frac{v^2}{4} - \frac{u^2}{9} = 1 . \tag{1}$$

This equation is considerably easier to handle, and is recognized as an equation of a hyperbola. You know something about hyperbolas, but we continue with our general approach so that after you have seen it work in familiar situations you may be able to use it in unfamiliar ones.

The graph is symmetric with respect to both new axes, and hence with respect to the origin. If we solve (1) for v in terms of u we get $v = \pm \frac{2}{3}\sqrt{u^2 + 9}$. This makes it clear that for a large, positive value of u , the two values of v are one large and positive, the other large and negative. (1) also shows that if (u,v) is any point on the graph, then $|v| \geq 2$. For

$\frac{u^2}{9} \geq 0$, and since $\frac{v^2}{4} - \frac{u^2}{9} = 1$, $\frac{v^2}{4} \geq 1$. Thus no point of the graph lies above $v = -2$ and below $v = 2$.

Now let us consider the part of the curve which lies in the first quadrant. For this we can use the equation

$$v = \frac{2}{3}\sqrt{u^2 + 9}$$

where $u \geq 0$. It seems almost obvious that when u is large, v is very nearly equal to $\frac{2}{3}u$. We can confirm this guess quite simply. Clearly $v > \frac{2}{3}u$, so let us consider $v - \frac{2}{3}u$, in the hope that we can prove it approaches 0 as u grows very large.

$$\begin{aligned} v - \frac{2}{3}u &= \frac{2}{3}\sqrt{u^2 + 9} - \frac{2}{3}u \\ &= \frac{2}{3}(\sqrt{u^2 + 9} - u) \\ &= \frac{2}{3}\frac{(\sqrt{u^2 + 9} + u)(\sqrt{u^2 + 9} - u)}{\sqrt{u^2 + 9} + u} \\ &= \frac{2}{3}\,\frac{u^2 + 9 - u^2}{\sqrt{u^2 + 9} + u} \\ &= \frac{6}{\sqrt{u^2 + 9} + u} . \end{aligned}$$

By taking large enough values of u we can make $v - \frac{2}{3}u$ as near to zero as we like. Thus we have shown that in the first quadrant, the graph lies above the line $v = \frac{2}{3}u$ but arbitrarily close to it for large enough u . In other words, $v = \frac{2}{3}u$ is an asymptote of the curve. By similar arguments we can show that $v = \frac{2}{3}u$ is also asymptotic to the part of the curve in the third quadrant, and that $v = -\frac{2}{3}u$ is asymptotic to the parts of the curve in the second and fourth quadrants.

The results above have been stated in terms of the new coordinates. They can easily be restated in terms of the old. For example, the asymptotes are the lines $y - 2 = \pm\frac{2}{3}(x + 1)$.

Finally we consider the intercepts. Setting $u = 0$ in (1) we get $\frac{v^2}{4} = 1$, so the v-intercepts are 2 and -2 . Setting $v = 0$ we get $-\frac{u^2}{9} = 1$, which has no solution. Hence the curve does not intersect the u-axis. The x- and y-intercepts can be found by the same sort of procedure, but since we are chiefly interested in sketching the curve, let's not bother with them.

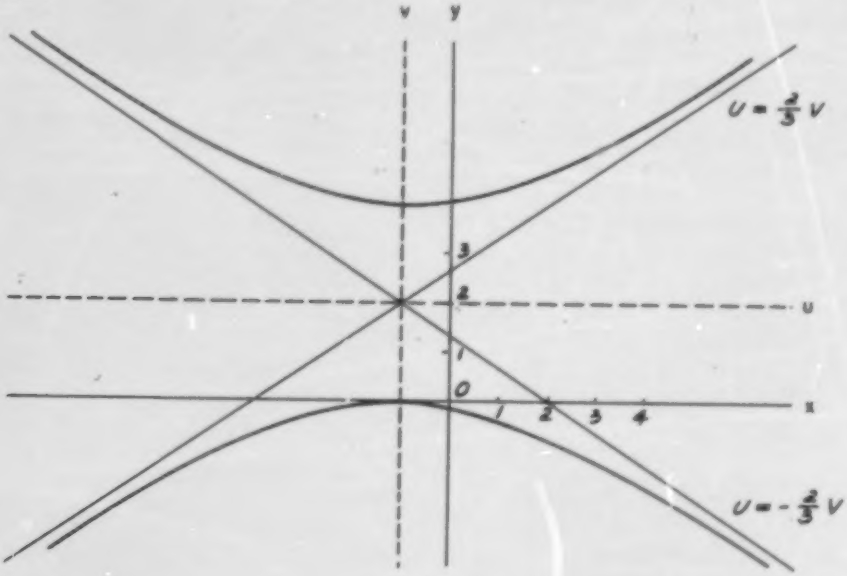


Figure 6-20

The hyperbola is sketched above. Notice that we can draw a fairly accurate graph without finding the coordinates of any points but the vertices. (What are the vertices of a hyperbola?)

When you first studied the hyperbola you learned that the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 .$$

This is an illustration of a principle which is sometimes useful in sketching loci. It can be expressed loosely in the following way. If $f(x,y) = g(x,y) \cdot h(x,y)$, the graph of $f(x,y) = 0$ is the union of the graphs of $g(x,y) = 0$ and $h(x,y) = 0$. Thus since

$$x^2 - y^2 - x + 5y - 6 = (x - y + 2)(x + y - 3)$$

the graph of

$$x^2 - y^2 - x + 5y - 6 = 0$$

is the pair of the lines which are the graphs of

$$x - y + 2 = 0$$

and

$$x + y - 3 = 0 .$$

Before trying to prove the principle we had better find out more accurately what it says. Let's "factor" $x + y$:

$$x + y = (x^2 - y^2) \cdot \frac{1}{x - y} .$$

Unfortunately, the graph of

$$x + y = 0$$

is a line, the graph of

$$x^2 - y^2 = 0$$

is two lines, while the graph of

$$\frac{1}{x - y} = 0$$

is the null set.

The difficulty lies in the notion of factoring. When we speak of factoring a positive integer, we mean expressing it as the product of two smaller positive integers. When we speak of factoring a polynomial, we mean expressing it as the product of two polynomials each of lower degree than the given polynomial and having coefficients of some specified type (say rational numbers). There is no such agreement as to what it means to factor an arbitrary function. For our present purposes it is enough to say that we have a factorization of $f(x,y)$ if, for every (x,y) in the domain of f,

$$f(x,y) = g(x,y) \cdot h(x,y) .$$

Of course, this allows uninteresting factorizations like

$$x^2 + y^2 = 1 \cdot (x^2 + y^2)$$

but it excludes the sort of thing that got us into trouble above, since $x + y$ is defined for every x and y, while $\frac{1}{x - y}$ is not defined if $x = y$.

With this interpretation of "factor" we can state the principle referred to above.

THEOREM 6-1. If $f(x,y)$ has the factorization

$$f(x,y) = g(x,y) \cdot h(x,y) .$$

The graph of $f(x,y) = 0$ is the union of the graphs of $g(x,y) = 0$ and $h(x,y) = 0$.

Proof: The point (a,b) is on the graph of

$$f(x,y) = 0$$

if, and only if,

$$f(a,b) = 0 .$$

But

$$f(a,b) = g(a,b) \cdot h(a,b)$$

and hence

$$f(a,b) = 0$$

if, and only if

$$g(a,b) = 0$$

or

$$h(a,b) = 0$$

that is, if, and only if, (a,b) lies on the graph of

$$g(x,y) = 0$$

or the graph of

$$h(x,y) = 0 .$$

Example 10. The graph of

$$(y - x + 2)(x^2 + 4y^2 - 2x + 16y + 13) = 0$$

is made up of the graph of

$$y - x + 2 = 0$$

and the graph of

$$x^2 + 4y^2 - 2x + 16y + 13 = 0 .$$

The former is a straight line. If we rewrite the equation of the latter in the form

$$\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{1} = 1$$

we see that it is an ellipse, with center $(1,-2)$, symmetric about the lines

$$x = 1$$

and

$$y = -2$$

and with major and minor axes of lengths 4 and 2, respectively. Both graphs are sketched below.

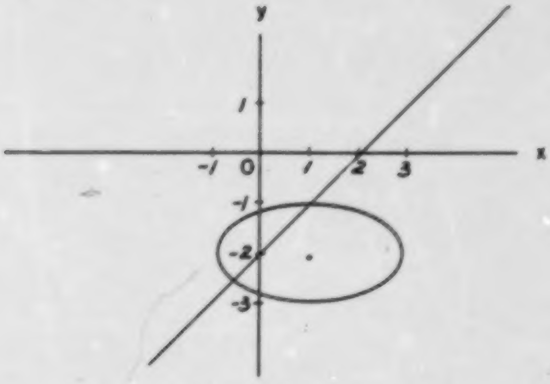


Figure 6-21

If we are given two parametric equations for a locus in a plane, there are two methods of sketching the locus (unless the equations are too complicated). We can eliminate the parameter between the two equations and graph the resulting equation in x and y, or we can choose some values of the parameter, compute the corresponding values of x and y, and draw a curve through the points thus determined. We illustrate both methods in the next example.

Example 11. Draw the graph of the parametric equations

$$x = 4t^2 - 2 , \quad y = 4t^4 . \tag{1}$$

Solution. First let's eliminate the parameter and graph the resulting equation. From the first equation we find that $2t^2 = \frac{x + 2}{2}$. Substituting this in the second equation gives

$$y = \frac{1}{4}(x + 2)^2 . \tag{2}$$

The graph of (2) is a parabola. It is sketched below.

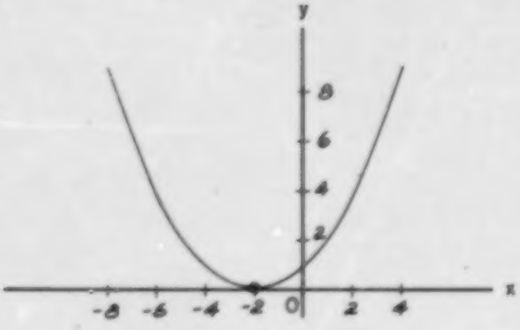


Figure 6-22

Now let's use the second method described above. The table below shows the results of our computations.

t	-2	-1	0	1	2
x	14	2	-2	2	14
y	64	4	0	4	64

We notice at once that we have found no values of x smaller than -2. It would be natural to jump to the conclusion that we had chosen the values of t foolishly, but that is not the explanation. Since $x = 4t^2 - 2$ and $4t^2 \geq 0$, it follows that $x \geq -2$ for every point on the graph. The trouble is that Equations (1) and (2) are simply not equivalent. The graph of (1) is half a parabola. It is the intersection of the graphs of (2) and the inequality $x \geq -2$. If you look back over our reasoning you will see it proves that the locus of (1) is contained in the locus of (2), but it does not prove they are identical.

Obviously the elimination of t was not as harmless an operation as it looked and we must study it more carefully. At a certain point we found from the first equation in (1) that $2t^2 = \frac{x+2}{2}$. Then we squared, getting the equation $4t^4 = \frac{(x+2)^2}{4}$. These two are not equivalent, since in the first, $x \geq -2$ while the second puts no restriction on x. This is no surprise since the same sort of thing comes up in the solution of equations involving radicals. In future we shall be careful not to square, or divide by zero, or do anything else of that sort when eliminating a parameter, and then perhaps we'll not get into trouble as we did above. Unfortunately it isn't that simple.

Example 12. What locus is represented by the parametric equations

(3) $$x = \sin t \quad y = \sin t \text{ ?}$$

Solution. Eliminating t in the only sensible way gives the equation $y = x$. The graph of this is a line, while the locus of (3) is the segment determined by $(-1,-1)$ and $(1,1)$. Equations (3) are an analytic condition for a segment stated without inequalities.

There is no simple way out of this difficulty, and we end our discussion with the warning that when you eliminate the parameter from a pair of parametric equations for a curve, you must then check to see whether the locus of the resulting equation is the locus of the original pair of equations.

The nature of the parameter may impose certain natural restrictions or bounds on the values of the variables involved. In some problems we may wish to impose such restrictions, and in that case we have, not a difficulty, but a special tool. It is important that we learn the uses and limitations of our tools, so that we do not try to use a screwdriver to drive nails.

All the analytic conditions we have considered so far in this section have been equations. Our last two examples deal with inequalities.

Example 13. Discuss and sketch the locus of the inequality

$$2x - 3y + 4 < 0 .$$

Solution. We shall use simple arguments about inequalities. Suppose (x_0,y_0) is on the line $2x - 3y + 4 = 0$, so that $2x_0 - 3y_0 + 4 = 0$. Now consider a point (x_0,y_1), with $y_1 > y_0$. Then $3y_1 > 3y_0$ and $2x_0 - 3y_1 + 4 < 2x_0 - 3y_0 + 4 = 0$. Thus (x_0,y_1) is a point of the locus. Similarly, if $y_2 < y_0$, $2x_0 - 3y_2 + 4 > 0$ and (x_0,y_2) is not a point of the locus. Thus any point directly above a point of the line is in the locus, while any point directly below a point of the line is not. Therefore the locus is the half-plane indicated below.

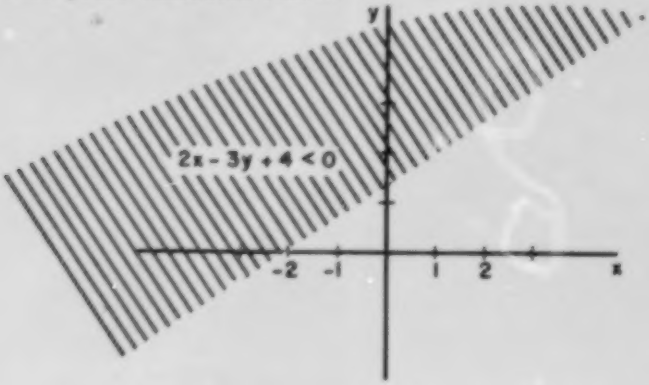


Figure 6-23

Example 14. Discuss and sketch the locus of the inequality

(4) $$2x^2 - 8x - y + 7 \geq 0 .$$

Solution. By completing the square we can rewrite this inequality in the form

$$2(x - 2)^2 - y - 1 \geq 0 .$$

Now suppose $2(x_0 - 2)^2 - y_0 - 1 = 0$. If $y_1 \leq y_0$ then $2(x_0 - 2)^2 - y_1 - 1 \geq 0$. Thus if (x_0,y_0) is on the graph of the equation

(5) $$2(x - 2)^2 - y + 1 = 0$$

and $y_1 \leq y_0$, we see that (x_0,y_1) is a point of our locus. By a similar argument we can show that if $y_2 > y_0$, then (x_0,y_2) is not a point of our locus. Thus our locus is the set of points below or on the parabola represented by Equation (5). It, or rather some of it, is shaded in the sketch below.

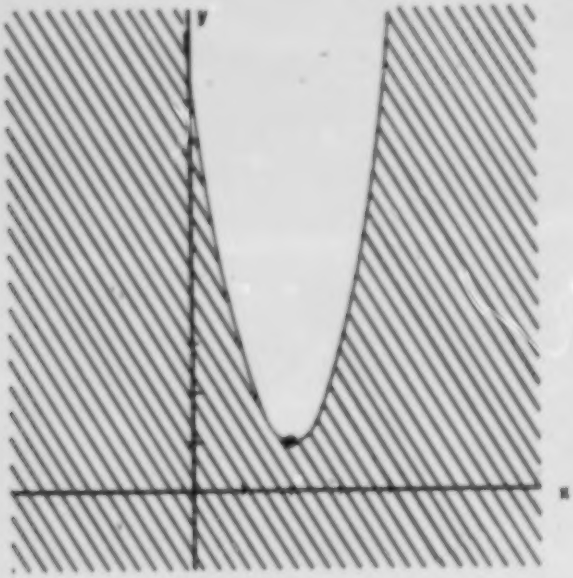


Figure 6-24

Exercises 6-3

In these exercises discuss and sketch the graphs of the conditions given. In your discussion you may find it useful to consider symmetry, extent, periodicity, intercepts, and asymptotes. When the condition is a pair of parametric equations, eliminate the parameter if you can, but be sure then to indicate any restrictions on the values of the variables.

1. $y = 2$
2. $y = -3$
3. $x = -1$
4. $x = 4$
5. $y = -x + 3$
6. $y = 2x - 1$
7. $x - 2y + 3 = 0$
8. $2x + 3y - 5 = 0$
9. $\frac{x}{2} - \frac{y}{3} = 1$

10. $\frac{x}{3} + \frac{y}{4} = 1$

11. $x = 1 - 2t$, $y = 2 + 3t$
12. $x = 2t$, $y = -2 - t$
13. $x^2 + y^2 - 4x + 2y + 4 = 0$
14. $x^2 + y^2 + 2x - 3 = 0$
15. $x^2 + y^2 + 2x - 2y + 2 = 0$
16. $y^2 = x(x - 2)(x - 3)$
17. $x^2 = (y + 1)(y - 1)(y - 4)$
18. $xy^2 - 2y - x = 0$
19. $y = \sin 2x$
20. $x = \sin y$
21. $y = 2 \sin x$
22. $x = \cos y$
23. $y = 1 + \cos x$
24. $y = \tan 2x$
25. $y = 2^x$
26. $y = 2^{-x}$
27. $y = 2^{x^2}$
28. $y = 3^{x^3}$
29. $y = \ln x$ (Note: This may also be written $y = \log_e x$.)
30. $y = \ln x^2$ (See above.)
31. $y = \log_2 x$
32. $x = t^2 + 1$, $y = 5t^2 + 4$
33. $x = \frac{1}{t}$, $y = 3t$
34. $x = 2 \cos \phi$, $y = 2 \sin \phi$
35. $x = 2 \cos \phi$, $y = 4 \sin \phi$
36. $x = 3 \cos^3 \phi$, $y = 3 \sin^3 \phi$
37. $x = \sin^2 \phi$, $y = \cos^2 \phi$
38. $x = \sec^2 \theta$, $y = \tan^2 \theta$
39. $y > x^2$
40. $\frac{x^2}{9} + \frac{y^2}{4} < 1$

41. $y^2 - 2x - 4y + 2 < 0$

42. $x^2 + y^2 + 4x + 6y + 9 \geq 0$

43. $y^2 = x^3$

44. $x^3 + xy^2 - 4y^2 = 0$

45. $x^3 + xy^2 - 3x^2 + y^2 = 0$

46. $x^2y + 4y - x = 0$

47. $x^4 + y^4 = a^4$

6-4. Graphs and Conditions (Polar Coordinates)

In this section we discuss the problem of sketching the graphs of analytic conditions in polar coordinates. The most important such conditions are equations, and we shall confine our attention to this case except for a few exercises.

The most straightforward way to draw the graph of an equation in polar coordinates is to plot a number of points of the locus and draw a curve through them. If the equation has the form $r = f(\theta)$, we can construct a table giving the values of r corresponding to a number of values of θ. No matter how many points we plot, there always remains the question of how the curve behaves elsewhere, that is, between the points we have plotted. If the equation is not too complicated, we can get a good deal of information by studying the functions involved.

As was the case for equations in rectangular coordinates, we can often get useful information about the curve by considering symmetry and extent. Asymptotes of curves given by equations in polar coordinates are not easy to find from the equations, and we shall not discuss the problem. However, if the curve has a fairly simple equation in rectangular coordinates, we may be able to find its asymptotes by studying that.

As you know, given a polar coordinate system in a plane, each point has infinitely many pairs of coordinates. This fact gives rise to certain difficulties that we have already met in Chapter 5 but we now consider them in greater detail. As in the previous section we shall develop additional theory and useful methods of approach in our discussion of a number of examples.

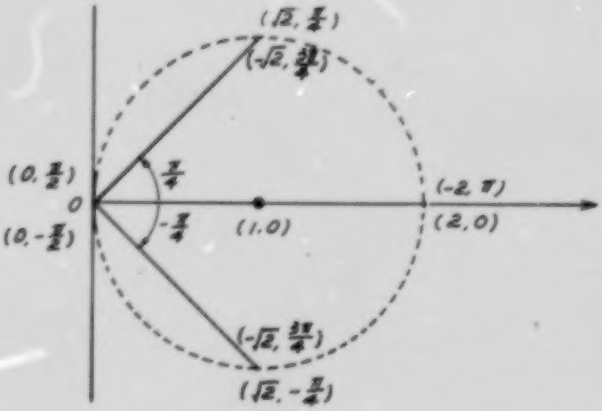
Example 1. Sketch and discuss the graph of the equation $r = 2 \cos \theta$.

Solution. Strictly speaking, we should state explicitly that r and θ are to be interpreted as polar coordinates. We shall not do so in the rest of this section, since there is no danger of ambiguity.

Since $|\cos \theta| \leq 1$ for all θ , the graph is bounded. Since $\cos(-\theta) = \cos \theta$ for all θ , if the point (r_0, θ_0) is on the graph, so is the point $(r_0, -\theta_0)$. Thus the graph is symmetric with respect to the line containing the polar axis. It is also symmetric with respect to the point $(1,0)$, but it is much easier to show this by using an equation in rectangular coordinates for the locus. The table below shows the values of r corresponding to several values of θ . The cosine function has period 2π , so any θ-interval of length 2π will do.

θ	0	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{2}$	$\pm\frac{3\pi}{4}$	π
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2

The graph is sketched below.



$(\sqrt{2}, \frac{\pi}{4})$
$(-\sqrt{2}, \frac{3\pi}{4})$
$(0, \frac{\pi}{2})$
0
$(0, -\frac{\pi}{2})$
$\frac{\pi}{4}$
$-\frac{\pi}{4}$
$(1,0)$
$(-2, \pi)$
$(2,0)$
$(-\sqrt{2}, \frac{3\pi}{4})$
$(\sqrt{2}, -\frac{\pi}{4})$

Figure 6-25

It looks like a circle (probably because it was drawn with a compass), but all we know so far, even if we make use of our knowledge of the cosine function, is that it is roughly circular.

That the graph really is a circle can be proved as follows. The graph of $r^2 = 2r \cos \theta$ is the same as the graph of $r = 2 \cos \theta$. For the only points that might be on the former but not on the latter are points with $r = 0$, and the origin, which is on the latter, is the only such point. If we take a rectangular coordinate system with its axes in the usual positions with respect to the polar axis, we find that the graph has the equation

$$x^2 + y^2 = 2x \, .$$

Example 2. Sketch and discuss the graph of the equation $r = \sin 3\theta$.

Solution. This graph, too, is bounded, since $|\sin 3\theta| \leq 1$ for all θ. Whether there is a point or line about which the graph is symmetric is not obvious from the equation, so we postpone the discussion of symmetry till we have sketched the graph. It will prove nothing but it will suggest what is probably true. The table below shows the values of r corresponding to a number of values of θ. If we needed a fairly accurate graph of the equation we would have to consider more values of θ, but since we know how $\sin 3\theta$ varies with θ, this table will do.

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
r	1	0	-1	0	1	0	-1	0	1	0	-1

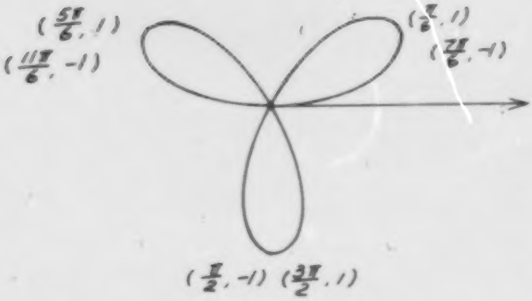


Figure 6-26

The sketch suggests there is symmetry about each of the lines $\theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$, and $\theta = \frac{3\pi}{2}$. Let us check the first of these conjectures. If we wish to compare $f(\frac{\pi}{6} + \alpha)$ and $f(\frac{\pi}{6} - \alpha)$ we obtain in the first case $r = \sin 3(\frac{\pi}{6} + \alpha)$ and in the second case $r = \sin 3(\frac{\pi}{6} - \alpha)$. These become $r = \sin(\frac{\pi}{2} + 3\alpha)$, and $r = \sin(\frac{\pi}{2} - 3\alpha)$; which in turn become $r = \cos 3\alpha$ and $r = \cos 3\alpha$. The identity of these equations establishes the symmetry we were checking. The same method can be used to deal with the other lines. The graph is not symmetric about any point, but we shall not prove this.

Example 3. Sketch and discuss the graph of the equation $r = 1 - 2 \sin \theta$.

Solution. Once more the graph is bounded, and we postpone the discussion of symmetry until below.

This time we shall sketch the graph without making a table, introducing first an auxiliary graph of a kind that is often useful in graphing polar equations. This auxiliary graph is the graph of the equation $y = 1 - 2 \sin x$, drawn on a plane with a rectangular coordinate system. We have learned to do this readily by the addition and multiplication of ordinates, as shown in Section 6-3, and illustrated below for the values $0 \leq x \leq 2\pi$. For the purpose of illustrating certain details of the discussion we will sometimes use different scales on the axes in the graphs in this section.

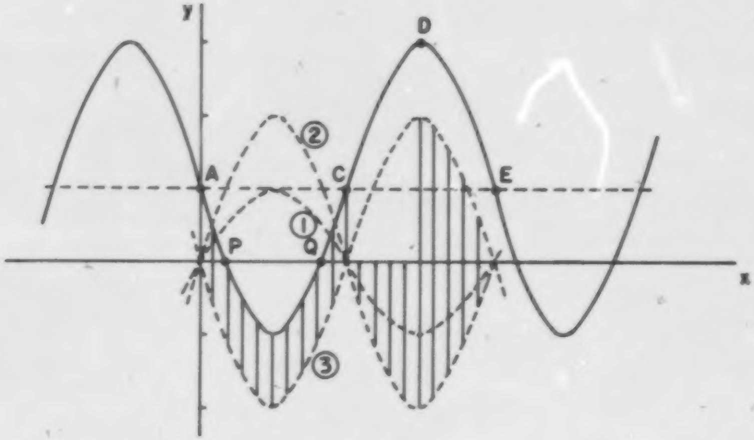


Figure 6-27

We suggest the following sequence:

(1) Sketch the familiar curve ① : $y = \sin x$.

(2) Expand ① away from the x-axis to get ② : $y = 2 \sin x$.

(3) Reflect ② in the x-axis to get ③ : $y = -2 \sin x$.

(4) Raise ③ 1 unit to get our graph: $y = 1 - 2 \sin x$.

We now use this graph of the equation $y = 1 - 2 \sin x$ to give us coordinates of points of the polar graph of $r = 1 - 2 \sin \theta$, and obtain the polar graph given in Figure 6-28.

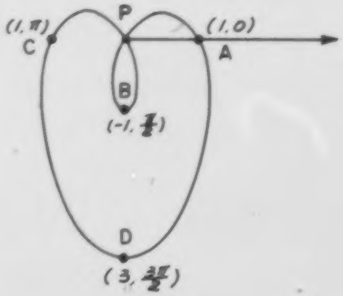


Figure 6-28

This curve is called a limaçon. We have indicated with the same letters corresponding points on the two graphs. Note that the lack of a unique polar representation of a point is shown in the fact that points P and Q of Figure 6-27 (and infinitely many more not shown) all correspond to point P of Figure 6-28. Also, points A and E of Figure 6-27 (and infinitely many more not shown) all correspond to point A of Figure 6-28. The inverted arch below the x-axis of Figure 6-27 corresponds to the small-inside loop of Figure 6-28.

Figure 6-28 suggests that the graph is symmetric about the line through the pole perpendicular to the polar axis, that is, the line for which one equation is $\theta = \frac{\pi}{2}$. We check this by comparing $f(\frac{\pi}{2} - \alpha)$ and $f(\frac{\pi}{2} + \alpha)$. In the first case $r = 1 - 2 \sin(\frac{\pi}{2} - \alpha)$ and in the second case $r = 1 - 2 \sin(\frac{\pi}{2} + \alpha)$. In both cases we obtain from familiar trigonometric relationships $r = 1 - 2 \cos \alpha$ which means that the two cases give equivalent equations, and the symmetry is proved.

Finally, the related polar equation is $r = -(1 - 2\sin(\theta + \pi)) = -(1 + 2\sin\theta)$. To show that the polar graph of this equation is the same limaçon as the one we obtained in Figure 6-28, we use a method similar to the method of addition of ordinates for graphs in rectangular coordinates. The method, called addition of radii, which may be new to you, is useful in sketching certain new graphs related to familiar ones.

We have seen earlier that the polar graph of $r = 2\sin\theta$ is a circle of radius 1, with its center at $(1, \frac{\pi}{2})$ indicated as ① in Figure 6-29(a). Consider a number of rays drawn from 0 to points of this circle, $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$, $\overrightarrow{OP_3}$, ... Find points Q_1, Q_2, Q_3, ... on these respective rays so that $d(P_1,Q_1) = d(P_2,Q_2) = d(P_3,Q_3), \ldots, = 1$, as shown in Figure 6-29(a), which shows the graph of $r = 1 + 2\sin\theta$.

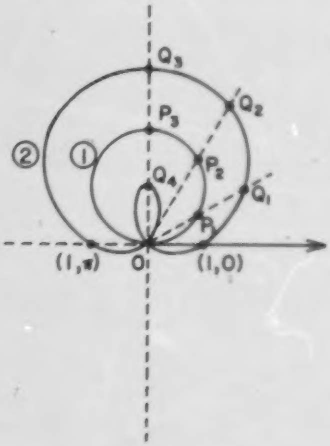


Figure 6-29(a)

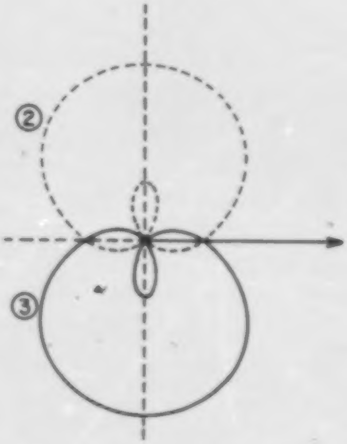


Figure 6-29(b)

Note that when $\pi < \theta < \frac{3\pi}{2}$ we have $0 > 2\sin\theta > -2$, therefore $1 > (1 + 2\sin\theta) > -1$, and the Q points of Figure 6-29(a) are on the right half of the inside loop of the graph. In the same way when $\frac{3\pi}{2} < \theta < 2\pi$ we get the rest of the inside loop.

Thus the locus of all the Q points is the graph marked ② which is a limaçon whose polar representation is $r = 1 + 2\sin\theta$. This process of using the P points to find the Q points and the graph ② is called the addition of radii.

Since we want the graph of $r = -(1 + \sin\theta)$ we now find the symmetric image of ② with respect to the pole. It is graph ③ which we recognize as the same limaçon as in Figure 6-28.

Example 4. Discuss and sketch the graph of the equation $r = \frac{1}{1 + \sin\theta}$.

Solution. This graph is not bounded, since r can be made arbitrarily large by picking θ so that $\sin\theta$ is sufficiently close to -1. By the method used in Examples 2 and 3, we find the graph is symmetric about the line $\theta = \frac{\pi}{2}$. It can be sketched from the table below.

θ	0	$\frac{\pi}{2}$	π	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	2π
r	1	$\frac{1}{2}$	1	3.4	7.5	Undefined	7.5	3.4	1

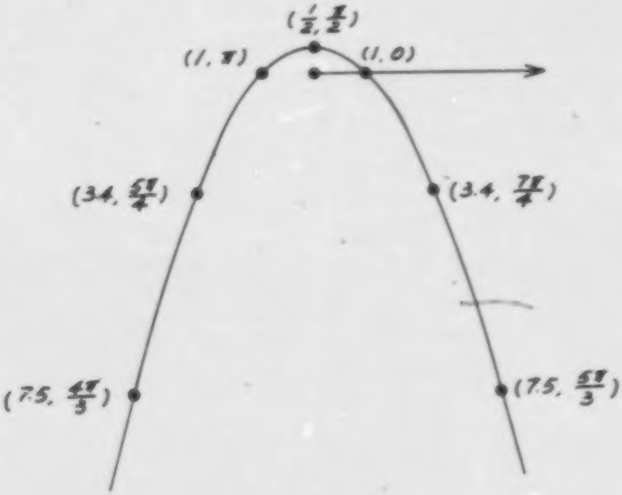


Figure 6-30

The sketch suggests the graph may be a parabola. That it is may be shown as follows. The equation

$$r = \frac{1}{1 + \sin\theta}$$

is equivalent to the equation

$$r + r \sin \theta = 1 .$$

If we introduce a rectangular coordinate system with its axes located as usual, the graph has the equation

$$\sqrt{x^2 + y^2} = 1 - y .$$

This is an equation of the parabola consisting of all points as far from the origin as they are from the line $y = 1$.

Exercises 6-4

In each of the exercises below, discuss and sketch the graph of the condition given. In your discussion, consider whatever geometric properties you can infer from the equations. Write the related polar equation for each. If you can, find a condition in rectangular coordinates for the same locus and identify the locus.

1. $r = 3$
2. $r = -2$
3. $\theta = \frac{\pi}{6}$
4. $\theta = -\frac{3\pi}{2}$
5. $r = 3 \sin \theta$
6. $r = \sin 2\theta$
7. $r = \cos 2\theta$
8. $r = \sin 5\theta$
9. $r \cos \theta = -3$
10. $r \cos (\theta - \frac{5\pi}{6}) = 3$
11. $r = \frac{3}{1 - \cos \theta}$
12. $r = \frac{9}{4 - 5 \cos \theta}$
13. $r = 2(1 + \sin \theta)$
14. $r = 2 \tan \theta$. (There are vertical asymptotes; try to find them.)
15. $r = \frac{4}{\theta}$
16. $r = 2 \cos \theta - 1$
17. $r = 2 - 3 \cos \theta$
18. $r = 2 + \sin \theta$

19. $r^2 = \cos 2\theta$

20. $r^2 = 4 \sin 2\theta$

21. $r = 4 \tan\theta \sec\theta$

22. $r = 2(1 + \sin^2\theta)$

23. $r = \dfrac{5}{1 + \cos\theta}$

24. $r \leq 2$

25. $|r| \leq 2$

26. $2 < r < 3$

27. $0 \leq \theta \leq \frac{\pi}{4}$

28. $0 \leq \theta \leq \frac{\pi}{4}$, $r \geq 0$

6-5. Intersections of Graphs (Rectangular Coordinates)

The intersection of two sets is the collection of objects that belong to both the sets. Now the graph of the equation $f(x,y) = 0$ is the set of points whose coordinates satisfy the equation, i.e. $\{(x,y) : f(x,y) = 0\}$. Hence the intersection of the graphs of $f(x,y) = 0$ and $g(x,y) = 0$ is the set of points whose coordinates satisfy both equations, i.e. $\{(x,y) : f(x,y) = 0$ and $g(x,y) = 0\}$. If f and g are linear functions, the intersection of the graphs of $f(x,y) = 0$ and $g(x,y) = 0$ is the set of points which lie on two lines, in other words the intersection of the two lines. In general, the intersection of the graphs of $f(x,y) = 0$ and $g(x,y) = 0$ is found by solving the two equations simultaneously.

Example 1. The intersection of the lines with equations $x - 2y - 1 = 0$ and $x + y = 2$ is the point $(\frac{5}{3}, \frac{1}{3})$.

Example 2. The intersection of the lines with equations $x - 2y - 1 = 0$ and $2x - 4y - 3 = 0$ is the null set. In other words, the lines are parallel.

Example 3. The intersection of the graphs of $y = \sin x$ and $y = \cos x$ is a bit harder to find. At each point (x,y) where the curves intersect we have $\sin x = \cos x$. Thus $x = \frac{\pi}{4} + k\pi$, where k is an integer. Then

$y = \frac{\sqrt{2}}{2}$ when k is even, $y = -\frac{\sqrt{2}}{2}$ when k is odd. This last statement can be written more compactly in a form frequently used by mathematicians:

$y = (-1)^k \frac{\sqrt{2}}{2}$, where k is an integer.

Example 4. The intersection of the graphs of $x - y + 3 \leq 0$ and $2x - xy + 4 \geq 0$ is the set of points on or above the line $x - y + 3 = 0$ and on or below the line $2x - y + 4 = 0$. It is the doubly shaded area in the figure below, and its boundary along parts of the lines.

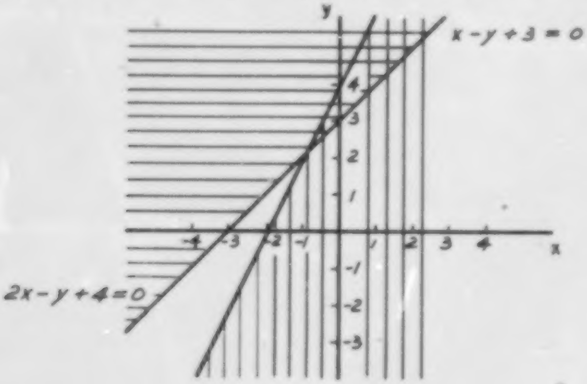


Figure 6-31

The problem of finding the intersection of two graphs can be very complicated, and we shall not spend much more time on it here. However, there is another example which is of interest.

Example 5. Find the intersection of $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 + 2x + 2y - 2 = 0$. We could consider the first equation as a quadratic equation in y and use the quadratic formula to express y in terms of x. We could get $y = 2 \pm \sqrt{8 + 2x - x^2}$. We could then substitute this in the second equation and solve for x. (Carry the work a bit further so you will appreciate the difficulties.)

This problem can be solved much more easily by using the principle of linear combination, which you studied in algebra. The system

(1)
$$\begin{aligned} x^2 + y^2 - 2x - 4y - 4 &= 0 \\ x^2 + y^2 + 2x + 2y - 2 &= 0 \end{aligned}$$

is equivalent to the system

$$(2)\qquad \begin{aligned} a(x^2+y^2-2x-4y-4)+b(x^2+y^2+2x+2y-2)&=0\\ x^2+y^2+2x+2y-2&=0 \end{aligned}$$

as long as $a \neq 0$. If $a = -1$ and $b = 1$, the second system becomes

$$(3)\qquad \begin{aligned} 4x+6y+2&=0\\ x^2+y^2+2x+2y-2&=0\,. \end{aligned}$$

Now the first equation in (3) is linear. Using it, we can express y in terms of x, substitute the result in the second equation, and have left nothing worse than a quadratic equation in x. The points of intersection are $(1,-1)$ and $(-\frac{23}{13},\frac{11}{13})$.

This solution has a geometric interpretation which is worth investigating.

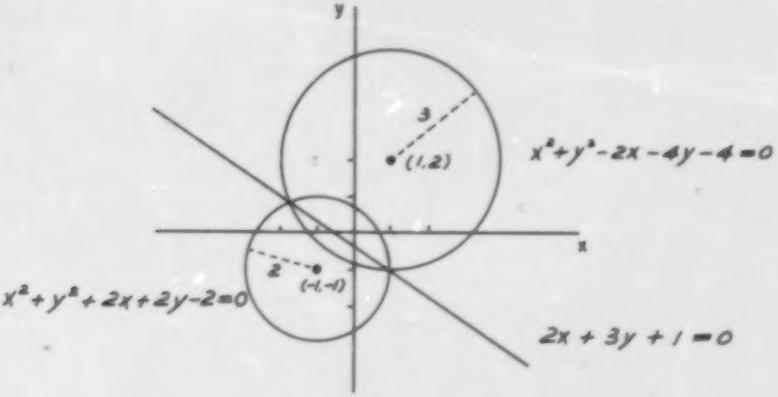


Figure 6-32

The graphs of the equations in (1) are circles. (How can you check this?) They are shown above. Now the graph of the first equation in (3) is a line and that equation is a special case of the first equation in (2). But if the coordinates of a point satisfy the two equations in (1), they clearly satisfy the first equation in (2), no matter what a and b are. Thus the graph of the first equation in (3) passes through all points of intersection of the two circles and must be the line containing the common chord, which is shown in the sketch above. If $a \neq -b$ which implies that a and b are not both zero, the first equation in (2) is that of a circle passing through the points of intersection of the two original circles. (As a matter of fact, each such circle may be obtained by some choice of a and b. Can you prove this?)

This result can be generalized. If $f(x,y) = 0$ and $g(x,y) = 0$ are equations of two loci, then the locus of $af(x,y) + bg(x,y) = 0$ contains the intersection of the two original loci. For suppose (x_0,y_0) lies on the original loci. Then $f(x_0,y_0) = 0$, $g(x_0,y_0) = 0$, and hence $af(x_0,y_0) + bg(x_0,y_0) = 0$. (This is true, though not very interesting, even when $a = b = 0$.)

Exercises 6-5

In each of the exercises below, find the intersection of the loci determined by the conditions given. Use both algebraic and geometric methods.

1. $x = 2$, $x - 2y = 2$
2. $x - y + 1 = 0$, $2x + y - 7 = 0$
3. $x + y - 1 = 0$, $2x + y = 0$
4. $x - 2y + 3 = 0$, $2x + y - 2 = 0$
5. $x - 2y + 3 = 0$, $2x - 4y + 5 = 0$
6. $x^2 + y^2 = 4$, $y = 2x$
7. $x^2 + y^2 = 2$, $x + y = 0$
8. $x^2 + y^2 - 2x + 4y + 5 = 0$, $3x + y - 1 = 0$
9. $x^2 + y^2 + 2x + 2y - 2 = 0$, $\frac{x}{4} + \frac{y}{5} = 1$
10. $y^2 = 4x$, $x - 2y + 3 = 0$
11. $4x^2 - 3y^2 = 1$, $x - y = 0$
12. $x^2 + 2y^2 = 4$, $x - y - 1 = 0$
13. $x^2 + y^2 = 11$, $x^2 + y^2 - 2x - 8 = 0$
14. $x^2 + y^2 = 15$, $2x^2 + y^2 = 24$
15. $xy^2 - xy - 4y + 4 = 0$, $y = x$
16. $x^2 + y^2 = 2$, $y = x^2$
17. $y - x^2 > 0$, $y - x - 1 < 0$

18. $x^2 + y^2 \leq 4$, $x - y^2 > 0$

19. $x + 2y + 3 < 0$, $3x - y + 5 > 0$, $2x - 3y + 1 < 0$

6-6. Intersection of Loci (Polar Coordinates)

In the previous section we discussed the intersection of loci given by equations in rectangular coordinates. The method we used works for loci determined by equations in polar coordinates, but, as we shall see, there are added complications. Let us take up first a simple case.

Example 1. Consider the graphs of $r = 1$ and $r = 2 \cos \theta$. They are the circles shown below.

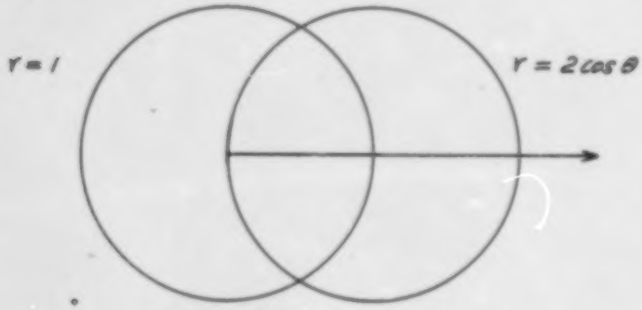


Figure 6-33

Solving the equations simultaneously we get $2 \cos \theta = 1$, $\cos \theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. (There are infinitely many other solutions of the equations, but since the sine and cosine functions have period 2π , we need consider only solutions with $0 \leq \theta < 2\pi$.) Of course, $r = 1$. This is consistent with our sketch.

Example 2. Now consider the equations $r = 2 \cos \theta$ and $r = 2 \sin \theta$. Once more their graphs are circles, which are shown in the figure below.

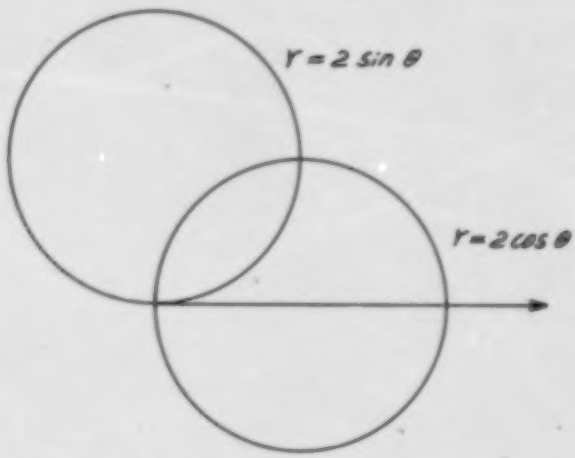


Figure 6-34

There appear to be two points of intersection. Let us solve the two equations simultaneously and compare our answer with the figure. Setting $2 \cos \theta = 2 \sin \theta$ we find $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. (As before, we need consider only solutions with $0 \leq \theta \leq 2\pi$.) The first gives $r = \sqrt{2}$, the second $r = -\sqrt{2}$. We have not, however, found the two points of intersection shown in the figure. We have found two sets of polar coordinates for the same point. This reminds us once more that while a rectangular coordinate system in a plane is a one-to-one correspondence between the points in the plane and the ordered pairs of real numbers, every point in the plane has infinitely many different pairs of polar coordinates.

This is also the source of our other difficulty. Clearly the pole lies on both curves, but our algebraic method did not find this intersection. The trouble is that the coordinates $r = 0$, $\theta = \frac{\pi}{2}$ satisfy the first equation but not the second, while the coordinates $r = 0$, $\theta = 0$ satisfy the second but not the first. Both pairs, of course, represent the pole, whose coordinates require special comment. If P is any point other than the pole, its coordinates, $(r, \theta + 2n\pi)$, allow infinitely many, but not all numbers as second coordinate. For the pole, however, the coordinates $(0, \theta)$ allow any number as a possible replacement for θ. Geometrically this means that, if there is any θ for which $r = f(\theta)$ becomes zero, the graph must contain the pole. We have already found in this example that $(0, \frac{\pi}{2})$ satisfies the first equation, and (0,0) the second, which means that the pole lies on both graphs and is therefore a point of intersection.

This leads to a small but important caution when finding intersections of polar graphs of $r = f(\theta)$, and $r = g(\theta)$. Check first to see if each graph contains the pole by seeing if there is any θ for which $r = f(\theta)$ equals zero, or any ϕ for which $r = g(\phi)$ equals zero. If both conditions can be satisfied, then, whether or not $\theta = \phi$, both graphs contain the pole, which is therefore an intersection point. Then you can proceed with the usual simultaneous solution of the two equations.

Example 3. Find the points of intersection of the graphs of

$$r = \frac{1}{2 + 2\cos\theta} \quad \text{and} \quad r = 2\cos\theta + 1 .$$

Solution. These graphs, which are related to some we have discussed earlier, are shown below. The pole is on the second graph but not the first, hence is not a point of intersection.

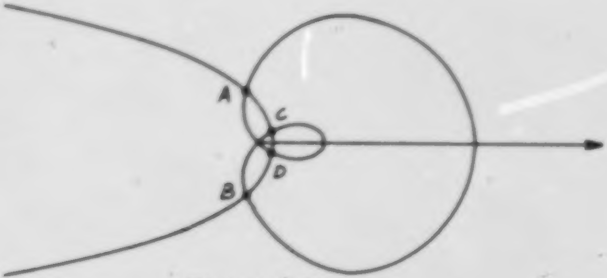


Figure 6-35

There appear to be four points of intersection.

Now let us solve the two equations simultaneously. Setting the expressions for r in the two equations equal to each other, we get

$$\frac{1}{2 + 2\cos\theta} = 2\cos\theta + 1 .$$

Simplifying, we get

$$4\cos^2\theta + 6\cos\theta + 1 = 0$$

from which we find that

$$\cos\theta = \frac{1}{4}(-3 \pm \sqrt{5})$$

or

$$\cos\theta \approx -1.31 \quad \text{or} \quad -.19 \ .$$

The first is a perfectly good root of the quadratic equation for $\cos\theta$, but it is not a possible value for $\cos\theta$. (Why not?) From a table of values of the trigonometric functions we find that if $\cos\theta \approx -.19$, then

$$\theta \approx 101^\circ \quad \text{or} \quad \theta \approx 259^\circ \ .$$

Then

$$r \approx .62 \ .$$

It is clear that we have found the points A and B of the figure, but what about C and D ? It is not too hard to guess the answer if we remember that a polar graph may have other analytic representations. In our algebraic solution we merely equated two of the infinitely many equivalent polar equations available for each curve. Fortunately we need not try them all; for the purposes of the course we can always find all the intersections of two polar graphs from the simultaneous solution of an equation of one of them with both of the related polar equations of the other. The limaçon $r = 2\cos\theta + 1$ has the related polar equation $r = -(2\cos(\theta+\pi) + 1)$ or $r = 2\cos\theta - 1$. If we now solve simultaneously the equations

$$r = \frac{1}{2 + 2\cos\theta} \quad \text{and} \quad r = 2\cos\theta - 1$$

we get the coordinates of points C and D in our figure. They turn out to be approximately, $(.30, 49^\circ)$ and $(.30, 311^\circ)$.

The difficulty is not a simple one, so we shall take another look at it. Consider:

$$\begin{cases} (.62, 101^\circ) \\ (-.62, 281^\circ) \end{cases} \qquad \begin{cases} r = 2\cos\theta + \text{[illegible]} \\ r = 2\cos\theta - 1 \end{cases}$$

We have two pairs of coordinates for the same point, and two equations for the same curve. The first pair of coordinates satisfies the first equation but not the second and the second pair of coordinates satisfies the second but not the first. This situation should occasion not anxiety but care, and is entirely consistent with our definition of the polar graph of an equation as the set of points each of which has some pair of coordinates that satisfy it.

Exercises 6-6

In each of the exercises below, find the intersection of the loci determined by the conditions given. Write the related polar equation for each, to make sure you find all points of intersection. Sketch both loci, as a check on your algebra.

1. $r = \dfrac{2}{1 + \cos\theta}$, $\theta = 30^\circ$
2. $r = \dfrac{4}{1 + \sin\theta}$, $\theta = 135^\circ$
3. $r = 2\cos\theta$, $r = 2\sin\theta$
4. $r = \cos\theta$, $r = 1 - \cos\theta$
5. $r = \cos\theta$, $r = \sin 2\theta$
6. $r = 1 - \sin\theta$, $4r\sin\theta = 1$
7. $r = 1 + \cos\theta$, $r = \dfrac{1}{1 - \cos\theta}$

6-7. Families of Curves.

In Section 6-5 we mentioned the collection of lines through the intersection of two lines and the collection of circles (and the line) through the intersections of two circles. These are examples of what are called families of curves. The collection of all circles in a plane and the collection of all tangents to a parabola are other examples. In this section we shall proceed a bit further with this topic.

If a and b are not both zero, then

$$a(x - y + 3) + b(3x - y + 7) = 0 \tag{1}$$

is an equation of a line through the intersection, P , of

$$x - y + 3 = 0 \quad \text{and} \quad 3x - y + 7 = 0 .$$

Can we choose a and b so that the line is vertical? Yes. For if we let $a = 1$ and $b = -1$, the equation becomes

$$-2x - 4 = 0$$

or

$$x = -2 .$$

This is one method you learned in algebra for solving pairs of linear equations in two unknowns. In a similar way we could find the horizontal line through the intersection, which is equivalent to finding the y-coordinate of P. It turns out that $P = (-2,1)$.

Every line through $(-2,1)$ may be obtained by picking a and b suitably. For the slope of (1), if it has one, is $\frac{a + 3b}{a + b}$. If $a = -b$, then (1) has no slope, a fact we noted above in case $a = 1$, $b = -1$. And for any real number m, a and b may be chosen so that

$$\frac{a + 3b}{a + b} = m .$$

(This is not obvious. Can you prove it?)

Let us look at this family of lines from another point of view. The line through $(-2,1)$ with slope m has an equation

$$y - 1 = m(x + 2) . \tag{2}$$

For each real value of m we get a line, and different values of m give different lines. Thus, (2) is almost the same family as (1), the only difference being that the line $x = -2$, since it has no slope, is not a member of (2).

Among the members of the family (2) there should be two which are tangent to the circle $x^2 + y^2 = 1$. (One of them is obvious, but let's solve the problem as though we did not know one answer.) Intuitively, it is clear that a tangent to a circle is a line which intersects the circle in only one point. Let us solve (2) simultaneously with the equation of the circle, and then try to pick m so that there is only one solution. From (2),

$$y = mx + 2m + 1 .$$

Substituting this in $x^2 + y^2 = 1$ we get

$$x^2 + (mx + 2m + 1)^2 = 1$$

or

$$x^2 + m^2x^2 + 4m^2 + 1 + 4m^2x + 2mx + 4m = 1$$

or

$$(1 + m^2)x^2 + (4m^2 + 2m)x + 4m^2 + 4m = 0$$

This quadratic will have only one root (that is, a double root) if, and only if, its discriminant is zero. The discriminant turns out to be $-4m(3m + 4)$, which is zero if, and only if, $m = 0$ or $m = -\frac{4}{3}$.

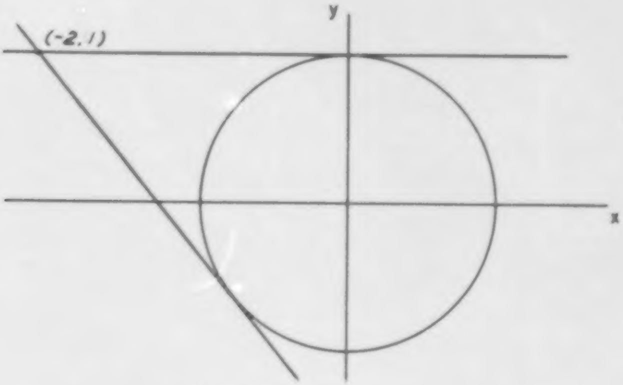


Figure 6-36

The figure shows the tangent lines for each case. Their equations are $y - 1 = 0$, and $4x + 3y + 5 = 0$.

Let us use the same method to find the family of tangents to the parabola $y = x^2$. Let (a,a^2) be any point on the parabola. The family of all but one of the lines through this point can be represented by the equation

$$y - a^2 = m(x - a).$$

(Which one is missing?) Expressing y in terms of a, m, and x, and substituting the result in the equation $y = x^2$, we get

$$x^2 - mx + ma - a^2 = 0.$$

This equation has a double root if, and only if, $m^2 - 4(ma - a^2) = 0$, i.e. if, and only if, $m = 2a$. Thus the slope of the tangent to $y = x^2$ at (a,a^2) is $2a$, and the family of lines tangent to the parabola can be represented by the equation

$$y - a^2 = 2a(x - a)$$

or, in somewhat simpler form

(3) $$y = 2ax - a^2 .$$

The "a" in (3) above is called a parameter. (The word was used earlier in the text in a different sense. That is, in a way, unfortunate, but both uses are very common.) It is difficult to define that word, but you must understand how "a" is used here. We might say "Let a be any real number. Then (3) is an equation of the tangent to $y = x^2$ at (a,a^2)." Here we are thinking of a as a fixed, but undetermined, real number. On the other hand, when we say that (3) represents the family of all tangents to the parabola $y = x^2$, we mean that each tangent to the parabola has an equation obtained by assigning a suitable real value to a, and each equation so obtainable is an equation of a tangent to the parabola. In other words, (3) is an ingenious way of writing infinitely many equations in a small space.

You have considered many other families of curves in earlier courses, whether you used this phrase or not. The equation $Ax + By + C = 0$ represents the family of all lines in a plane. The equation $y = mx + b$ represents the family of all lines which have slopes, that is, all lines which are not perpendicular to the x-axis. The equation $xy = k$ represents the family of all rectangular hyperbolas with the coordinate axes as their asymptotes (and the two axes themselves, obtained by setting $k = 0$ and sometimes called a degenerate hyperbola). The equation $(x - h)^2 + (y - k)^2 = r^2$ represents the family of all circles in a plane (and the point (h,k), obtained by setting $r = 0$ and sometimes called a point circle).

Sometimes it is useful to consider a family of curves and select from it those which have some additional property. For example, at one point in the discussion above we considered the family of lines which pass through a point of $y = x^2$, and then selected from this family the member having the additional property of being tangent to the parabola. Let's consider an analogous problem.

The family of all the circles in the plane can be represented by the equation

$$(x - h)^2 + (y - k)^2 = r^2 .$$

The center of each such circle is at (h,k). Which members of the family are tangent to both axes? If a circle is tangent to both axes its center is on the line $y = x$ or on the line $y = -x$. The family of circles with centers on the line $y = x$ can be represented by the equation

$$(x - h)^2 + (y - h)^2 = r^2 .$$

Such a circle will be tangent to both axes if, and only if, $r = |h|$ or $r^2 = h^2$. Thus the family of circles lying in the first or third quadrant and tangent to both axes can be represented by the equation

$$(x - h)^2 + (y - h)^2 = h^2 .$$

An equation representing those in the second or fourth quadrant can be found in a similar way.

Exercises 6-7

In each of the first 13 exercises, find an equation representing the family of curves described.

1. All vertical lines.
2. All horizontal lines.
3. All nonvertical lines through $(2,-1)$.
4. All nonvertical lines.
5. All circles with center $(-1,2)$.
6. All circles with radius 4 .
7. All parabolas with vertices at the origin and axes horizontal.
8. All lines parallel to $3x - 4y + 5 = 0$.
9. All lines perpendicular to $2x + y - 3 = 0$.
10. All lines tangent to the circle $x^2 + y^2 = 25$.
11. All lines that do not meet the circle $x^2 + y^2 = 25$.
12. All circles of radius 6 which go through the origin.
13. All circles of radius 1 such that the origin is not a point of the circle or its interior.
14. Find an equation of the line through the intersections of the lines $x - y + 6 = 0$ and $2x - y = 0$ and having x-intercept equal to 3 .
15. Find an equation of the line through the intersection of $x + y - 4 = 0$ and $2x - y + 8 = 0$ and having slope 1 .

16. Find an equation of the line passing through the intersection of the lines $x + y + 1 = 0$ and $x - 3y + 2 = 0$, and having no slope.

17. Find an equation of the line through the intersection of the lines $x - 2y + 3 = 0$ and $x + 3y - 2 = 0$ and the point $(1,1)$, without finding the intersection of the two lines.

18. Find an equation of the family of circles through the intersections of the circles $x^2 + y^2 - 2x - 35 = 0$ and $x^2 + y^2 + 2x + 4y - 44 = 0$, without finding the intersections of the two circles.

19. Find an equation of the line through the intersection of the lines $2x + 5y - 10 = 0$ and $3x - y + 19 = 0$ and perpendicular to the second of these lines.

20. Find an equation of the line through the intersection of $x + y - 4 = 0$ and $x - y + 2 = 0$ and parallel to $3x + 4y + 7 = 0$.

21. Find equations of all lines passing through the intersection of $5x - 2y = 0$ and $x - 2y + 8 = 0$ and cutting from the first quadrant triangles whose areas are 36.

22. Find equations of all lines through the intersection of $y - 10 = 0$ and $2x - y = 0$ which are 5 units from the origin.

6-8. Summary.

We have explored in some detail in this chapter the relations between the geometric properties of a set of points and the algebraic properties of its analytic representation. It was convenient to discuss the geometric properties under the headings of symmetry, extent, periodicity, intercepts, and asymptotes. We paid particular attention to the special situations that arise in polar coordinates from the lack of uniqueness in the correspondence between points and their polar coordinates, and the consequent lack of uniqueness in the correspondence between curves and their analytic representations.

Our discussion considered relationships between graphs and their conditions, first in rectangular and then in polar coordinates. We developed several useful techniques, notably the method of sketching a graph by addition and multiplication of ordinates in rectangular graphs, and by addition of radii in polar graphs.

These techniques were then applied to pairs of graphs and their intersections, and the corresponding pairs of analytic representations and their simultaneous solutions. We investigated in some detail the difficulties that arise here with polar coordinates and found the concept of related polar equations particularly useful in these cases.

Our consideration of more than two graphs at a time was confined to collections of graphs related by some common feature. These are called families of graphs, and we developed some useful concepts in defining such a family, and then selecting a particular member of it to fit some special requirement.

In our next chapter we sharpen our focus and discuss particularly a certain classification of graphs and their equations. These, the conic sections, have a valid claim to our special attention, both because they have been extensively studied for over 2000 years and because they have important and interesting application in many aspects of our lives today.

Chapter 6 - Review Exercises

1. Find the locus of the midpoint of all segments parallel to the x-axis, and terminated by the lines $x + y - 8 = 0$, $2x - y - 1 = 0$.

2. Find the locus of the midpoint of all segments parallel to the y-axis and terminated by the lines $x + y - 8 = 0$, $2x - y - 1 = 0$.

3. If $A = (-4,0)$ and $B = (4,0)$ find an equation for the locus of $P = (x,y)$ if:

   (a) $d(P,A) = 2d(P,B)$;
   (b) $d(P,A) + d(P,B) = 10$;
   (c) $d(P,A) - d(P,B) = 2$;
   (d) $\overleftrightarrow{PA} \perp \overleftrightarrow{PB}$;
   (e) slope of $\overleftrightarrow{PA}$ = twice the slope of $\overleftrightarrow{PB}$;
   (f) slope of $\overleftrightarrow{PA} = 1 +$ slope of $\overleftrightarrow{PB}$;
   (g) measure of $\angle APB = 45^\circ$;
   (h) sum of the measures of $\angle A$ and $\angle B$ is 120° ;
   (i) area of $\triangle ABP = 20$;
   (j) $d(P,A) < d(P,B)$.

4. The circle whose equation is $x^2 + y^2 = 36$ contains the point $A = (6,0)$. If $P = (x,y)$ is any other point of the circle, find an equation for the locus of the midpoints of $\overline{AP}$.

5. The circle whose equation is $x^2 + y^2 = 25$ contains the point $B = (0,5)$. If $Q = (x,y)$ is any other point of the circle, find an equation for the locus of points P such that Q is the midpoint of $\overline{BP}$.

6. The circle whose equation is $x^2 + y^2 = 100$ contains the point $C = (-10,0)$. A line through C meets the circle again at D, and the line $x = 20$ at E. Find an equation for the locus of the midpoint of $\overline{DE}$, for all positions of the line through C.

7. Find an equation for the locus of the midpoints of all chords of the circle $x^2 + y^2 - 4x + 8y = 0$ which are parallel to the line $y = 3x + 5$.

8. Find an equation for the line containing the midpoints of all chords of the ellipse $x^2 + 9y^2 = 36$ which are parallel to the line $x + y = 10$.

9. Find equations for the families of curves described below:
   (a) All lines which, with the polar axes, form a triangle whose area is 12.
   (b) All lines, the sum of whose intercepts is 6.
   (c) All circles tangent to the y-axis.
   (d) All circles tangent to the x-axis.
   (e) All circles with radius 1 that are tangent to the line $4x + 3y - 2 = 0$.
   (f) All circles tangent to the line $4x + 3y - 2 = 0$.
   (g) All circles of radius 6 such that the origin is an interior point.
   (h) All circles which go through the origin.
   (i) All circles which go through the point $(12,5)$.
   (j) All circles whose interior contain the origin.
   (k) All circles of radius 5, such that the origin is not a point of the circle or its interior.
   (l) All circles of radius d which are tangent to the line $ax + by + c = 0$.
   (m) All circles tangent to the lines $3x - 4y + 5 = 0$ and $4x - 3y + 9 = 0$.
   (n) All circles tangent to the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.
   (o) All circles which intersect or touch the x-axis.

(p) All circles which do not intersect or touch the y-axis.

(q) All circles which do not intersect or touch the line $ax + by + c = 0$.

(r) All circles in the interior of $x^2 + y^2 = 100$.

(s) All circles which intersect or touch the circle $x^2 + y^2 = 1$.

(t) All lines which intersect or touch the circle $x^2 + y^2 = 1$.

(u) All circles in the interior of the triangle determined by the points $O = (0,0)$, $A = (10,0)$ and $B = (0,10)$.

(v) All circles whose interiors contain the points A, B, and O of the previous exercise.

(w) All circles which are tangent internally to $x^2 + y^2 = 100$.

(x) All circles which are tangent externally to $x^2 + y^2 = 100$.

(y) All circles to which the circle $x^2 + y^2 = 100$ is tangent internally.

(z) All circles tangent to the line $ax + by + c = 0$ and passing through the point (r,s).

10. Sketch the graphs of the following conditions.

(a) $|x| = 3$

(b) $|y + 2| = 7$

(c) $|y| < 5$

(d) $|x - 3| \leq 4$

(e) $x^2 + y^2 > 1$

(f) $x^2 < y$

(g) $|x| < |y|$

(h) $|x| + |y| = 6$

(i) $x^2 < x + 20$

(j) $y^2 > 3y$

(k) $xy + 2x > y + 2$

(ℓ) $xy + 3x + 4y > -12$

(m) $5x - 2y + 10 > xy$

(n) $xy = 3y - x + 3$

(o) $3x + 2y - 6 < xy$

(p) $x^3 + xy^2 = 9x$

(q) $x^3y + xy^3 = xy$

(r) $(x - 3)^2 = (y - 5)^2$

(s) $y = \sqrt{x}$

(t) $x = \sqrt{36 - y^2}$

11. Sketch the graphs of the following pairs of parametric equations.

(a) $\begin{cases} x = t + 1 , \\ y = t^2 + 2 . \end{cases}$

(b) $\begin{cases} x = \frac{1}{t} , \\ y = t^2 . \end{cases}$

(c) $\begin{cases} x = 2t - 3 , \\ y = 3 - 2t . \end{cases}$

(d) $\begin{cases} x = t + 1 , \\ y = \sin t . \end{cases}$

(e) $\begin{cases} x = t^2 , \\ y = \cos t^2 . \end{cases}$

(f) $\begin{cases} x > t , \\ y = 2t . \end{cases}$

(g) $\begin{cases} x < t , \\ y = t + 1 . \end{cases}$

(h) $\begin{cases} x > 2t , \\ y = t^2 . \end{cases}$

(i) $\begin{cases} x > t , \\ y < t . \end{cases}$

(j) $\begin{cases} x < t , \\ y > t^2 . \end{cases}$

12. Sketch and discuss the polar graphs of the following conditions.

(a) $r = \cos 2\theta$

(b) $r = \cos(\theta + 2)$

(c) $r = \sin(\theta - \frac{\pi}{2})$

(d) $r = 2 \sin 3\theta$

(e) $r = 3 \sin 2\theta$

(f) $r = 1 + \sin \theta$

(g) $r = 2 - \cos \theta$

(h) $r = 1 + 2 \sin \theta$

13. Sketch the graphs of $y = x^2$ and $y = x^4$ with respect to the same axes. Generalize.

14. Sketch the graphs of $y = x$, $y = x^3$ and $y = x^5$ with respect to the same axes. Generalize.

15. Sketch the graph of $y = 3 \sin x + 4 \cos x$. What does it remind you of? Note that this equation can also be written in the form

$$y = 5(\tfrac{3}{5} \sin x + \tfrac{4}{5} \cos x) \text{ and that } (\tfrac{3}{5})^2 + (\tfrac{4}{5})^2 = 1 .$$

Finally, use these facts and a well known trigonometric identity to write a third form of the original equation.

16. Generalize the result of the preceding exercise by considering the equation $y = a \sin x + b \cos x$, where a and b are arbitrary real numbers.

17. Prove analytically that if a set of points in a plane is symmetric with respect to each of two mutually perpendicular lines, it is symmetric with respect to their intersection.

18. Prove that the graph of the pair $x = at + b$, $y = f(t)$ of parametric equations is identical with the graph of the equation $y = f(\frac{x - b}{a})$ obtained by eliminating t in the natural way. Thus there are cases in which it is possible to eliminate a parameter without getting into trouble.

19. Make a graph of $y = a + b \sin (cx + d)$ for each of the following sets of values of a , b , c , d .

(a) $a = 2$, $b = 3$, $c = 2$, $d = \frac{\pi}{2}$.

(b) $a = -3$, $b = 2$, $c = -3$, $d = \pi$.

(c) $a = 3$, $b = -2$, $c = 2$, $d = \frac{3\pi}{2}$.

(d) $a = -2$, $b = 2$, $c = 3$, $d = 0$.

Challenge Exercises

1. Sketch the rectangular graph of $y = \sin 4x \sin x$. Discuss the graph of $y = (6 + \sin x) \sin 12x$, and generalize suitably. Consider $y = \sin 1000\pi t \cdot \sin 1000000\pi t$, which is related to equations which describe *amplitude modulation*, in radio broadcasting.

2. For discussion and experiment, if an oscilloscope is available. Adjust the controls to get a stationary sine wave on the screen, then alter one control at a time to change the amplitude, the wave-length, the frequency, etc. If available and possible, find the constants of the oscilloscope and write the actual equations of the curve.

Chapter 7

CONIC SECTIONS

7-1. Introduction

This chapter is intended to give you a better understanding of the curves called conic sections. When you studied geometry, you investigated properties of a circle. In your study of algebra you worked with equations of the various conic sections and their properties. Here we shall first consider briefly the history of conic sections. Then we shall give a formal definition of a conic section and use polar coordinates to obtain a standard polar equation of a conic section. We shall see how equations in polar form are related to the equations in rectangular form that you have already studied. We shall derive properties of these curves and work with some of their many applications.

In studying conic sections you will use the knowledge and techniques acquired so far in analytic geometry. Both rectangular and polar coordinates will be used; often parametric representation will be helpful. Ideas of locus and curve sketching will be used.

It is assumed that you have studied the definitions, equations, and properties of the conic sections; brief summaries will show you what you are expected to know. If you find that you need more detail, you will find it in the following sections of Intermediate Mathematics:

6-3. The Parabola (pages 315-321)
6-4. The General Definition of the Conic (pages 326-331)
6-5. The Circle and the Ellipse (pages 333-336)
6-6. The Hyperbola (pages 342-348)

7-2. History and Applications of the Conic Sections

The curves called conic sections were so named after their historical discovery as intersections of a plane and a surface called a right circular cone. A right circular cone is the surface generated by a line moving about a circle and containing a fixed point on the normal to the plane of the circle at the center of the circle. The fixed point, called the *vertex*, separates the surface into two parts called *nappes*. Each line determined by the vertex and a point of the circle is called an *element* of the cone. The normal to the plane of the circle containing the vertex is called the *axis* of the cone. The proper conic sections are circles, ellipses, parabolas, and hyperbolas.

The discovery of the conic sections is attributed to the Greek mathematician Menaechmus (circa 375-325 B.C.), who was a tutor to Alexander the Great. He apparently used them in an attempt to solve three famous problems, the trisection of an angle, the duplication of a cube, and the squaring of a circle. Although the Greek mathematicians were primarily interested in the mathematical applications of the conic sections, they did know some of the optical properties of the curves. The definition of the conic sections which we shall use is attributed to Apollonius, who flourished before 200 B.C.

Further discoveries of the physical applications of the conic sections did not occur until the conjectures of the German scientist and mathematician Johannes Kepler (1571 - 1630), who hypothesized that the planets moved in elliptic orbits with the sun as a focus. The theoretical development of Kepler's conjectures followed the gravitation theory and calculus developed by Isaac Newton (1642 - 1727). In fact, it may be shown that any physical object subject to a force which is described by what is called an inverse square law will move in an orbit which is a conic section. Gravity is such a force; the electrical force between charged bodies was found to be another such force by Charles Augustin de Coulomb (1736 - 1806).

Today we find applications of the theory of conic sections in the orbits of planets, comets, and artificial satellites. The theory also applies to the lenses of telescopes, microscopes, and other optical instruments, weather prediction, communication by satellites, geological surveying, and the construction of buildings and bridges. Conics also occur in the study of atomic structure, the long range guidance systems for ships and aircraft, the location of hidden gun emplacements and the detection of approaching enemy ships and aircraft. The surfaces of revolution formed by the conic sections, which will be considered in Chapter 9, find application in the sciences dealing with light, sound, and radio waves.

It is helpful to visualize the four conic sections formed by the intersections of a plane and a right circular cone. We illustrate the physical possibilities below.

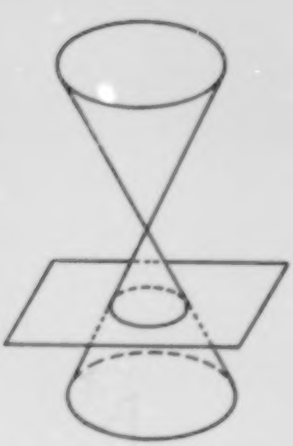


Figure 7-1a: Circle

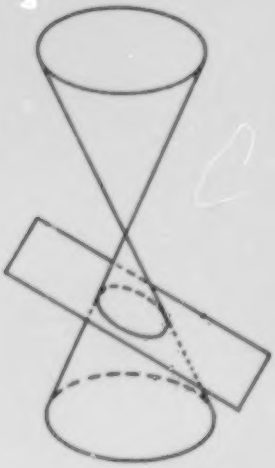


Figure 7-1b: Ellipse

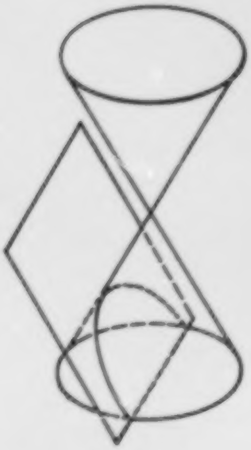


Figure 7-1c: Parabola

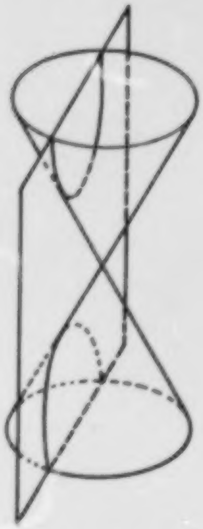


Figure 7-1d: Hyperbola

A circle (Figure 7-1a) is the intersection of a cone and a plane perpendicular to the axis of the cone. An ellipse (Figure 7-1b) is the intersection of a cone and a plane which forms an acute angle with the axis. The measure of this acute angle is greater than the measure of the angle formed by the axis and an element of the cone. A parabola (Figure 7-1c) is the intersection of a cone and a plane parallel to an element of the cone. A hyperbola (Figure 7-1d) is the intersection of a cone and a plane which forms an angle with the axis whose measure is less than the measure of the angle formed by the axis and an element of the cone. These descriptions suggest that circles and ellipses are the sections formed when planes cut every element of the cone; parabolas are formed when planes cut some elements in one nappe of a cone; hyperbolas are formed when planes cut some elements in both nappes of the cone. Although the drawings of Figure 7-1 are limited, cones are infinite in extent; what is illustrated is only part of the parabola or hyperbola.

For a more complete and systematic geometric development of the conic sections, leading to the definition to be given in the following section, see Supplement to Chapter 7.

7-3. The Conic Sections in Polar Form

We shall choose as a defining characteristic of the conic sections that geometric property which leads most readily to their analytic description. This property relates all the conic sections except the circle.

> DEFINITIONS. A conic section is the locus of points in a plane such that for each point the ratio of its distance from a given point F in the plane to its distance from a given line D in the plane is a given constant e. The given point F is called a focus or focal point of the conic section. The given line D is a directrix of the conic section. The given constant e is the eccentricity of the conic section. If $0 < e < 1$, the conic section is called an ellipse. If $e = 1$, the conic section is called a parabola. If $e > 1$, the conic section is called a hyperbola.
>
> A circle is also a conic section and is the locus of points at a given distance from a given point. The given distance is called the radius of the circle and the given point is called the center of the circle.

In some ways it is simpler to describe the conic sections in polar coordinates. We are already familiar with the polar equation, or equation in polar coordinates, of a circle with center at the origin as $r = k$, where k is the radius.

We shall assume that the focal point does not lie on the directrix. Let the focus of the conic section be at the pole and let the directrix be perpendicular to the polar axis. Let the polar axis be oriented away from the directrix; that is, the ray that is the polar axis does not intersect the directrix. Let p be the distance from the pole to the directrix and let $P = (r,\theta)$ be a point of the conic section.

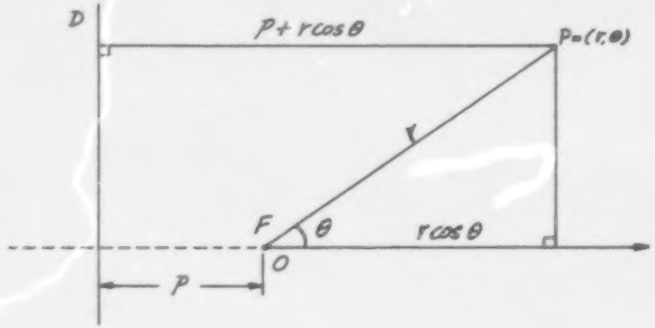


Figure 7-2

Then the distance from P to the focal point is r , and the distance from P to the directrix is $p + r\cos\theta$. Thus,

$$\frac{r}{p + r\cos\theta} = e\ .$$

Expressing r in terms of θ , we obtain

$$r = \frac{ep}{1 - e\cos\theta}\ . \tag{1}$$

In the above discussion we have assumed that the focal point did not lie on the directrix. If it does, we obtain certain figures which are called degenerate conics. Geometrically, they are the intersections of cones and planes containing the vertex of the cone. (For a more complete discussion, see Supplement to Chapter 7.)

If the focal point is on the directrix, then $p = 0$, and we may not perform certain algebraic operations, since division by zero would be indicated. We may express the analytic condition as follows:

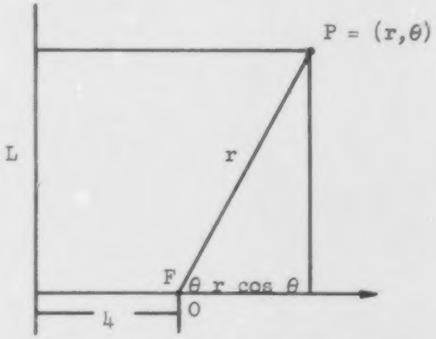
$$r = er\cos\theta\ .$$

If $0 < e < 1$, we have $r < r\cos\theta$, which is never true. If $e = 0$, we have $r = 0$, which is an equation of the pole. This is sometimes called a point-circle. (It is sometimes convenient to think of a circle as a special case of the ellipse. This is not consistent with our approach here, but it suggests why one may encounter the description of this locus as a point-ellipse.)

If $p = 0$ and $e = 1$, we obtain $r = r \cos \theta$. From this we may infer either $r = 0$, or $1 = \cos \theta$. The graph of $r = 0$ has just been discussed. The graph of $1 = \cos \theta$ is the line containing the polar axis; this we call a degenerate parabola. If $p = 0$ and $e > 1$, the equation $r = er \cos \theta$ will be satisfied when $\cos \theta = \frac{1}{e}$. Thus the locus is two distinct lines through the pole and is called a degenerate hyperbola. (There will be further discussion of degenerate conics in the Supplement to Chapter 7.)

Thus far we have considered the equation of a conic only in the case in which the focus is at the pole, the directrix is perpendicular to the polar axis, and the polar axis is oriented away from the directrix. Certain other cases will be considered in Example 2 and the exercises, but we shall not take up the case in which the directrix is oblique to the polar axis until we have studied rotation of the axes in Chapter 10.

Example 1. A fixed point F is 4 units from a given line L. Write an equation for the locus of points equidistant from P and L.



Solution. We place the pole of our polar coordinate system at F, and the polar axis perpendicular to L and directed away from L. Then for any point $P = (r,\theta)$ on the locus,

$$r = 4 + r \cos \theta ,$$

which becomes

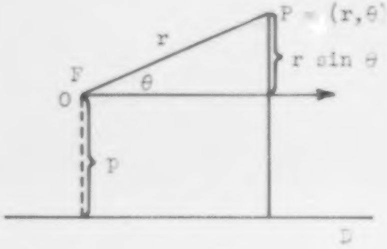
$$r = \frac{4}{1 - \cos \theta} .$$

This equation is in the form of Equation (1), and represents a parabola.

Example 2. What is a polar equation of a conic section with focus at the pole and directrix parallel to the polar axis and p units below it?

Solution. Let $P = (r,\theta)$ be a point of the curve. Then the distance from P to the focal point is r, and the distance from P to the directrix is $p + r \sin\theta$. Thus,

$$\frac{r}{p + r\sin\theta} = e .$$

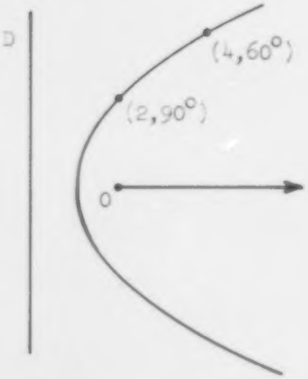


Expressing r in terms of θ, we obtain

$$r = \frac{ep}{1 - e\sin\theta} .$$

Example 3. Graph $r = \frac{2}{1 - \cos\theta}$.

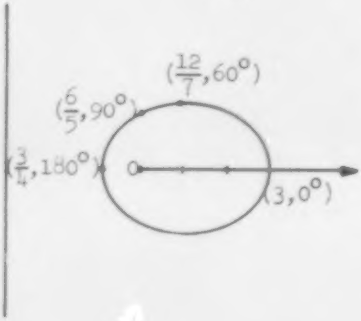
Solution. This equation is in the form of Equation (1) with $e = 1$, $p = 2$. Hence its graph is a parabola with focus at O, and directrix D perpendicular to the polar axis and 2 units to the left of the pole. The vertex must be midway between O and D. Location of one or two more points --say $(4,60^\circ)$ and $(2,90^\circ)$ --and use of symmetry then permit making a sketch.



Example 4. Graph $r = \frac{6}{5 - 3\cos\theta}$.

Solution. To obtain the form of Equation (1), we divide numerator and denominator of the fraction by 5, and write the numerator as the product of e and a number which must be p. We obtain

$$r = \frac{\frac{3}{5} \cdot 2}{1 - \frac{3}{5}\cos\theta} .$$



Since $e = \frac{3}{5}$ and $p = 2$, the graph is an ellipse; one vertex divides the normal segment joining the focus to the directrix in the ratio 3 to 5. We obtain a few more points--say $(3,0^\circ)$, $(\frac{12}{7},60^\circ)$, and $(\frac{6}{5},90^\circ)$ --and use symmetry to complete the graph.

Exercises 7-3

Graph each of the following:

1. $r = \frac{4}{1 - \cos\theta}$
2. $r = \frac{6}{2 - 2\cos\theta}$
3. $r = \frac{4}{2 - \cos\theta}$
4. $r = \frac{6}{3 - \cos\theta}$
5. $r = \frac{12}{4 - 5\cos\theta}$
6. $r = \frac{24}{2 - 6\cos\theta}$
7. $r = \frac{4}{1 - \sin\theta}$
8. $r = \frac{6}{2 - 2\sin\theta}$
9. What is a polar equation of a conic section with focus at the pole and directrix parallel to the polar axis and p units above it?
10. What is a polar equation of a conic section with focus at the pole and directrix perpendicular to the polar axis and p units to the right of the pole?
11. Using the results of Exercises 9 and 10, graph the following:

    (a) $r = \frac{4}{1 + \cos\theta}$

    (b) $r = \frac{12}{4 + 5\sin\theta}$

    (c) $r = \frac{8}{4 + 3\sin\theta}$

    (d) $r = \frac{10}{5 + 3\cos\theta}$

In Exercises 12-19, rewrite the equations in a form convenient for graphing, identify the conic section, and sketch the graph.

12. $r - 6 - r \cos \theta = 0$
13. $r - 10 - r \sin \theta = 0$
14. $3r - 12 - 2r \cos \theta = 0$
15. $3r - 12 - 4r \cos \theta = 0$
16. $r = 2 + r \sin \theta$
17. $r = 3 - 2r \cos \theta$
18. $\cos \theta = 1 - \frac{3}{r}$
19. $\sin \theta = \frac{r + 2}{r}$

20. An artificial satellite has the center of the earth as its focus. For a polar coordinate system in the plane of its orbit the distance of the satellite from the center of the earth at $\theta = 180^\circ$ is 5000 mi. and at $\theta = 90^\circ$ is 6000 mi. Assuming that the axis is along the line $\theta = 0^\circ$, find the equation describing the orbit and the greatest distance of the satellite from the center of the earth.

7-4. Conic Sections in Rectangular Form

We have developed polar equations for the conic sections in certain specified positions. For a circle with center at the pole, we have

$$r = k .$$

For the other conic sections with focus at the pole, and directrix perpendicular to the polar axis and p units to the left of the pole, we have

$$r = \frac{ep}{1 - e \cos \theta}, \qquad \text{representing} \begin{cases} \text{a parabola if } e = 1, \\ \text{an ellipse if } 0 < e < 1, \\ \text{a hyperbola if } e > 1 . \end{cases}$$

We shall find the corresponding rectangular equations by using the following equations, developed in Section 2-4:

$$x = r \cos \theta \qquad r^2 = x^2 + y^2$$

$$y = r \sin \theta \qquad \tan \theta = \frac{y}{x}, \ x \neq 0 .$$

Circle: If

$$r = k ,$$

then

$$r^2 = k^2 .$$

(This is equivalent to multiplying the members of $r - k = 0$ by the corresponding members of $r + k = 0$. Since these are both equations of the same circle, the graph of the resulting equation is the same as that of the original equation.)

Since $$r^2 = x^2 + y^2 ,$$

we may write $$x^2 + y^2 = k^2 .$$

We now consider the general equation

$$(1) \qquad r = \frac{ep}{1 - e \cos \theta} .$$

We multiply both members of the equation by $1 - e \cos \theta$ to obtain

$$r - er \cos \theta = ep$$

or $$r = e(r \cos \theta + p) ,$$

and square both members of the latter equation to obtain

$$(2) \qquad r^2 = e^2(r^2 \cos^2 \theta + 2pr \cos \theta + p^2) .$$

(Whenever we square both members of an equation we must be careful of the interpretation of the result. We have in effect multiplied both members of $r - e(r \cos \theta + p) = 0$ by the corresponding members of $r + e(r \cos \theta + p) = 0$. We recall from Section 5-2 that $r - e(r \cos \theta + p) = 0$ has the related polar equation

$$-r - e\big((-r) \cos (\theta + \pi) + p\big) = 0 .$$

Since $\cos (\theta + \pi) = - \cos \theta$, this is equivalent to

$$-r - e(r \cos \theta + p) = 0$$

or (3) $$r + e(r \cos \theta + p) = 0 .$$

Since the "factors" of Equation (2) are equivalent to Equation (1) and its related polar equation, it has the same graph as Equation (1). We may now proceed with the original discussion.) Using $r^2 = x^2 + y^2$ and $r \cos \theta = x$, we have

$$(4) \qquad x^2 + y^2 = e^2(x^2 + 2px + p^2) .$$

We now have our equation in rectangular coordinates and wish to examine it for the different values of e.

Parabola: Since $e = 1$, Equation (4) becomes

$$x^2 + y^2 = x^2 + 2px + p^2$$

or

$$y^2 = 2p(x + \frac{p}{2}).$$

This equation, as you may recognize from your study of algebra, represents a parabola with focus at the origin and vertex at $(-\frac{p}{2}, 0)$.

Example 1. Write in rectangular form and sketch the graph of

$$r = \frac{6}{1 - \cos\theta} .$$

Solution. The given equation yields $r - r\cos\theta = 6$, which after transformation becomes

$$\sqrt{x^2 + y^2} - x = 6$$

or

$$\sqrt{x^2 + y^2} = x + 6 .$$

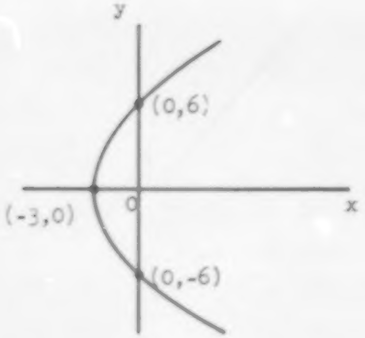
Therefore $x^2 + y^2 = x^2 + 12x + 36$,

and finally

$$y^2 = 12x + 36$$

or

$$y^2 = 12(x + 3) .$$



Ellipse: Here $0 < e < 1$. We rewrite Equation (4) as

$$x^2 + y^2 = e^2x^2 + 2e^2px + e^2p^2 .$$

We rearrange the terms to obtain

$$(1 - e^2)x^2 - 2e^2px + y^2 = e^2p^2 .$$

Since we are looking for a form that we can recognize as the equation of a conic that has a center, we use the technique of completing the square. Dividing by the coefficient of x^2 , we have

$$x^2 - 2\frac{e^2p}{1 - e^2}x + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2}$$

or $$x^2 - 2\frac{e^2p}{1 - e^2}x + \left(\frac{e^2p}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2} + \left(\frac{e^2p}{1 - e^2}\right)^2$$

or $$\left(x - \left(\frac{e^2p}{1 - e^2}\right)\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2} + \left(\frac{e^2p}{1 - e^2}\right)^2 .$$

Since $0 < e < 1$, $e^2 < 1$ and $1 - e^2 > 0$. Thus the coefficients of x^2 and y^2 are both positive. Although the equation above is quite cluttered with constants, it should be apparent that it has the form of the equation of an ellipse with center at $\left(\frac{e^2p}{1 - e^2}, 0\right)$.

Example 2. Write in rectangular form:

$$r = \frac{6}{1 - \frac{1}{2}\cos\theta} .$$

Solution. The given equation yields $r - \frac{1}{2}r\cos\theta = 6$, which, by substitution, becomes

$$\sqrt{x^2 + y^2} - \frac{1}{2}x = 6 .$$

Therefore $$\sqrt{x^2 + y^2} = \frac{1}{2}x + 6 ;$$

hence $$x^2 + y^2 = \frac{1}{4}x^2 + 6x + 36 .$$

Finally, this becomes $3x^2 + 4y^2 - 24x - 144 = 0$, which you may recognize as an equation for an ellipse in rectangular form. We may write this in standard form thus:

$$3(x^2 - 8x + 16) + 4y^2 = 144 + 48 ,$$

or $$3(x - 4)^2 + 4y^2 = 192 ,$$

or $$\frac{(x - 4)^2}{64} + \frac{y^2}{48} = 1 .$$

You may recognize that this equation represents an ellipse with center at $(4,0)$.

Hyperbola: The algebraic manipulation involved in expressing the equation of a hyperbola in rectangular form is identical with that for the ellipse. However, when we reach the form

$$\left(x - \left(\frac{e^2p}{1 - e^2}\right)\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2p^2}{1 - e^2} + \left(\frac{e^2p}{1 - e^2}\right)^2 ,$$

we note that since $e > 1$, $e^2 > 1$ and $1 - e^2 < 0$. Thus the coefficients of x^2 and y^2 have opposite signs.

It should be apparent that this is the equation of a hyperbola with center at $\left(\frac{e^2p}{1 - e^2}, 0\right)$.

Exercises 7-4

For each of the polar equations below you are asked to do three things:

(a) Sketch the graph.

(b) Write a corresponding equation in rectangular coordinates.

(c) Write the related polar equation.

1. $r = 3$
2. $r = 9$
3. $r = 2 \cos \theta$
4. $r = \cos \theta + \sin \theta$
5. $r = \frac{4}{1 - \cos \theta}$
6. $r = \frac{3}{1 + \cos \theta}$
7. $r = \frac{3}{1 - 2 \cos \theta}$
8. $r = \frac{6}{2 - \cos \theta}$
9. $r = \frac{5}{3 - 2 \cos \theta}$
10. $r = \frac{5}{2 - 3 \cos \theta}$
11. $r = 1 + \cos \theta$
12. $r - r \sin \theta = 2$
13. $4r - 3r \cos \theta = 12$
14. $4r + 5r \sin \theta = 20$

7-5. The Parabola

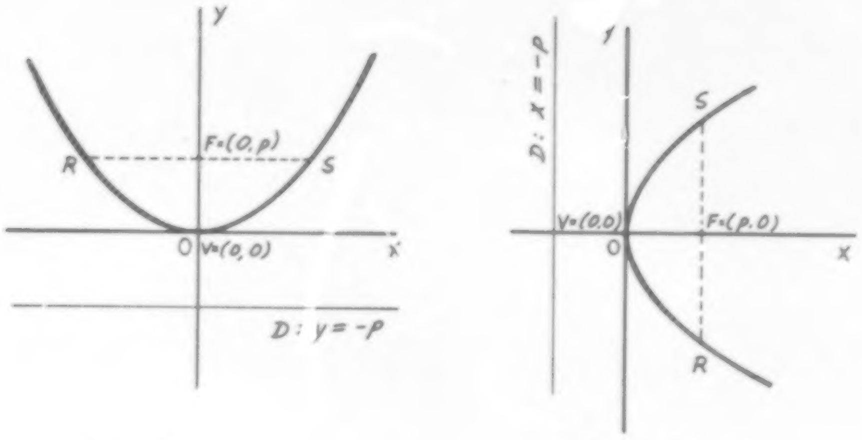
In this section and the following three we consider the four main kinds of conic sections: parabola, circle, ellipse, and hyperbola. There are brief summaries of the important definitions and properties. Equations in rectangular coordinates--often called standard forms--are given for these curves with axes on or parallel to the coordinate axes. Much of this information is not new; it is placed here because of its importance, and for your convenience.

The parabola is defined as the set of points equidistant from a fixed point (the focus) and a fixed line (the directrix). A parabola is symmetric with respect to the line through the focus perpendicular to the directrix. This line of symmetry is called the axis of the parabola, and its point of intersection with the parabola is called the vertex of the parabola.

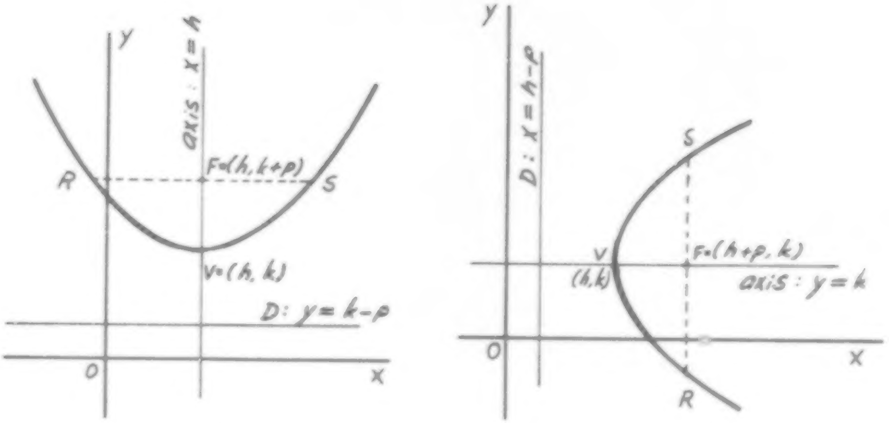
In Figure 7-3 F, V, and D indicate the focus, vertex, and directrix, respectively, and $|p|$ is the distance between F and V. If $p > 0$, the parabola extends upward or to the right as shown; if $p < 0$, it extends downward or to the left.

In making a quick sketch of a parabola, it is convenient, after locating V, F, and D, to find the length of the latus rectum. This is the chord of the parabola through the focus perpendicular to the axis. If in Equation (a) Figure 7-3 we set $y = p$, we find $x = \pm 2p$; thus, the length of the latus rectum is $|4p|$. (The student should verify that for each of the other standard forms of the equation given in Figure 7-3 the length of the latus rectum is also $|4p|$.)

In general a conic section has been defined as the set of points P such that the ratio of the distance from P to a fixed point, to the distance from P to a fixed line, is a constant e, called the eccentricity. For the parabola $e = 1$.


y
F=(0,p)
R
S
O
V=(0,0)
x
D: y = -p
D: x = -p
y
S
V=(0,0)
F=(p,0)
O
x
R

(a) $x^2 = 4py$ (c) $y^2 = 4px$


y
axis : x = h
F=(h,k+p)
R
S
V=(h,k)
D: y = k-p
O
x
y
D: x = h-p
S
V
(h,k)
F=(h+p,k)
axis : y = k
O
x
R

(b) $(x-h)^2 = 4p(y-k)$ (d) $(y-k)^2 = 4p(x-h)$

Figure 7-3

Our definition of the parabola makes no restriction on the position of the fixed point and line. What if the point is on the line? Our knowledge of geometry tells us that the locus must be the line perpendicular to the directrix at the fixed point. If we let $p = 0$ in, say, Equation (d) of Figure 7-3, we obtain

$$(y - k)^2 = 0 .$$

This equation represents a straight line. This locus is often called a degenerate parabola.

The parabola has important geometric properties, some of which concern tangents; these you will be able to derive more easily when you have studied calculus. One of the best known is the reflective property: light rays parallel to the axis of a parabolic reflector are concentrated at the focus, and light rays emanating from the focus are reflected parallel to the axis. This property, although usually illustrated in two dimensions, has more interest and physical applications in three dimensions. Such parabolic reflectors are used not only for light rays, but also for heat, sound, and microwaves. You may have seen such reflectors used with microphones, or radar antenna, or as parts of artificial satellites.

The parabola is also important in analyzing trajectories; the path of a projectile can be approximated by a parabola. Under certain conditions of loading, the cable of a suspension bridge hangs in the form of a parabola. Arches of bridges sometimes have parabolic form.

Example 1. Rewrite the equation $x^2 + 4x + 8y - 4 = 0$ in standard form. Write the coordinates of the vertex and focus and the equations of the axis and directrix.

Solution. Since x^2 is the only second-degree term, we group the x-terms and complete the square.

$$x^2 + 4x = -8y + 4$$

is equivalent to $$x^2 + 4x + 4 = -8y + 8 ,$$

or $$(x + 2)^2 = -8(y - 1) .$$

This last form we may compare with $(x - h)^2 = 4p(y - k)$, and recognize as an equation of the parabola with axis parallel to the y-axis, and vertex $(-2,1)$. Since $p = -2$, the parabola opens downward. The axis is a vertical line through the vertex; hence its equation is $x = -2$. The directrix is a horizontal line 2 units above the vertex and has the equation, $y = 3$. The focus, $(-2,-1)$, is two units below the vertex.

Example 2. Write an equation of the parabola with vertex $(3,2)$ and directrix $x = -1$.

Solution. Since the directrix is vertical, the axis is horizontal; an equation will be in the form (d) of Figure 7-3. The distance from V to the directrix is p ; here $p = 4$. Thus an equation is

$$(y - 2)^2 = 16(x - 3) .$$

Exercises 7-5

1. Rewrite each of the following equations in standard form; write the coordinates of vertex and focus, and equations of axis and directrix; draw the graph.

   (a) $x^2 = -16y$

   (b) $y^2 = 16x$

   (c) $5x^2 - 3y = 0$

   (d) $y^2 - 5y + 6x - 16 = 0$

   (e) $2x^2 - 8x - 3y + 11 = 0$

   (f) $y = ax^2 + bx + c$

2. We have noted that a special or degenerate case of the parabola occurs when the fixed point is on the fixed line. In this case Equation (d) of Figure 7-3 becomes $(y - k)^2 = 0$; the locus is a straight line parallel to the x-axis.

   (a) Find the degenerate case of each of the other standard forms of the equation of the parabola, and state what the locus is.

   (b) If a parabola is a section of a cone by a plane parallel to an element of the cone, can you explain these "degenerate parabolas" as limiting cases?

3. Derive an equation of a parabola to fit each of the following conditions by using the locus definition of a parabola.

   (a) Focus $(-1,-2)$, directrix $x = 2$

   (b) Focus $(-1,3)$, directrix $y = 2$

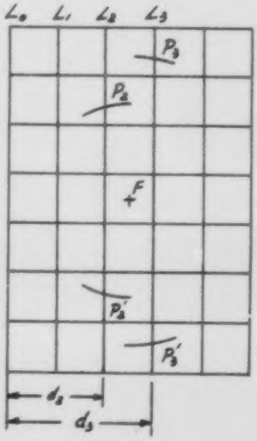
   (c) Vertex $(0,0)$, focus $(-5,0)$

   (d) Vertex $(4,5)$, directrix $x = 3$

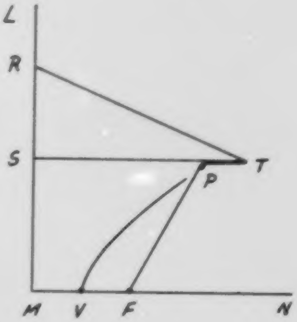
4. Obtain an equation for each of the parabolas for which conditions are given in Exercise 3 by using the standard forms of the equations.

5. Find an equation of a parabola to fit each of the following conditions.
   (a) Vertex $(0,0)$, directrix $2x - 5 = 0$
   (b) Vertex $(2,-3)$, directrix the x-axis
   (c) Vertex $(0,0)$, axis of symmetry the x-axis, passing through the point $(2,7)$
   (d) Latus rectum 16 , open down, vertex $(-2,3)$

6. Cross-section paper and a compass can be used as follows. Mark one of the printed lines L_0 and mark successive parallel lines L_1 , L_2 , ... Select any point F on the same side of L_0 as L_1 . With a compass measure on one of the printed lines the distance d_2 from L_0 to L_2 . With d_2 as radius and F as center, locate points P_2 and P_2' on L_2 . In a similar fashion, using d_3 as radius, locate P_3 and P_3' on L_3 . Prove that the points P_2 , P_2' , ... lie on a parabola.



7. To construct a parabola mechanically, place a straight edge L perpendicular to the line MN . Attach one end of a piece of string of length ST to point T of right triangle RST , and the other end to a point F on MN . With a pencil, hold the string against the side ST of the triangle as the side SR slides along ML . Prove that the point P of the pencil describes a parabola as the triangle slides.



Challenge Problems

1. In Section 6-7 it was shown that the family of tangents to the parabola $y = x^2$ at any point $P = (a,a^2)$ on the parabola can be represented by the equation $y = 2ax - a^2$. Prove the reflective property of the parabola for this case. (Hint: Show that the tangent makes equal angles with the line from P to the focus and the line through P parallel to the axis of the parabola.)

2. Again using the results of Section 6-7, prove the following statements for the parabola $y = x^2$.

   (a) The points of tangency of two perpendicular tangents are collinear with the focus.

   (b) The locus of the intersections of pairs of perpendicular tangents is the directrix.

7-6. The Circle

A circle is the set of points in a plane each of which is at a given distance from a fixed point of the plane. If the fixed point, called the center, is $C = (h,k)$, and the given distance is r, for the required set of points $P = (x,y)$ we have

$$(x - h)^2 + (y - k)^2 = r^2 .$$

If $r = 0$, the solution set is the single point (h,k); such a locus is often called a point-circle. If $r^2 < 0$, the solution set is the empty set; in this case the locus is sometimes said to be an imaginary circle.

Since there are three arbitrary constants h, k, r in the standard equation of a circle, it is in general possible to impose three geometric conditions on a circle. The following example will illustrate this.

Example 1. Find an equation of the circle which passes through the three points $(1,2)$, $(-1,1)$, $(2,-3)$.

Solution A. Using the equation $x^2 + y^2 + Dx + Ey + F = 0$, we write in turn the condition that each of the given points satisfies the equation.

$$1 + 4 + D + 2E + F = 0 \text{ , or } D + 2E + F = -5$$

$$1 + 1 - D + E + F = 0 \text{ , or } -D + E + F = -2$$

$$4 + 9 + 2D - 3E + F = 0 \text{ , or } 2D - 3E + F = -13$$

We now have a system of 3 equations in 3 unknowns; solving these by any desired method, we find that

$$D = -\frac{23}{11}, \qquad E = \frac{13}{11}, \text{ and } \qquad F = -\frac{58}{11} .$$

We substitute these values in the equation and multiply by 11 to obtain

$$11x^2 + 11y^2 - 23x + 13y - 58 = 0 .$$

Solution B. Here we use the condition that the center (h,k) is equidistant from any two points of the circle. We select the first two points and write this condition.

$$(h - 1)^2 + (k - 2)^2 = (h + 1)^2 + (k - 1)^2 \text{ , or } 4h + 2k = 3$$

We then do the same thing for the last two points.

$$(h + 1)^2 + (k - 1)^2 = (h - 2)^2 + (k + 3)^2 \text{ , or } 6h - 8k = 11$$

The coordinates of the center of the desired circle must satisfy both of these equations; solving them, we have

$$C = (h,k) = (\frac{23}{22}, -\frac{13}{22}) .$$

Now we find the radius r, the distance between C and any of the given points, say the first.

$$r^2 = (1 - \frac{23}{22})^2 + (2 + \frac{13}{22})^2$$

$$= \frac{(-1)^2 + (57)^2}{22^2}$$

$$r = \frac{1}{22}\sqrt{3250}$$

Thus the equation of the circle is

$$(x - \frac{23}{22})^2 + (y + \frac{13}{22})^2 = \frac{3250}{22^2} .$$

The student should satisfy himself that this equation, when simplified, is the same as the one obtained in Solution A. What happens to the solution of this problem if the three points are collinear?

Example 2. What is the locus of $36x^2 + 36y^2 - 36x + 48y + 24 = 0$?

Solution. We regroup the terms and apply the distributive law to obtain

$$36(x^2 - x) + 36(y^2 + \frac{4}{3}y) = -24 .$$

We complete the squares by adding the same numbers to each member of the equation, obtaining

$$36(x^2 - x + \frac{1}{4}) + 36(y^2 + \frac{4}{3}y + \frac{4}{9}) = -24 + 9 + 16 ,$$

which is equivalent to

$$(x - \frac{1}{2})^2 + (y + \frac{2}{3})^2 = \frac{1}{36} .$$

Hence the locus is a circle with center $(\frac{1}{2}, -\frac{2}{3})$, and radius $\frac{1}{6}$.

Exercises 7-6

1. Rewrite the following equations to show what each locus is; if it is a circle, find the center and radius.

   (a) $x^2 + y^2 - 8x = 0$

   (b) $x^2 + y^2 - 6x - 10y + 33 = 0$

   (c) $x^2 + y^2 - 4x + 8y + 20 = 0$

   (d) $x^2 + y^2 + 14x - 9y + 60 = 0$

   (e) $x^2 + y^2 - x + y = 0$

   (f) $x^2 + y^2 - 2ax - 2by + a^2 + b^2 = 0$

   (g) $5x^2 + 5y^2 - 6x + 4y + 2 = 0$

   (h) $2x^2 + 2y^2 - 2ax + 2by - ab = 0$

2. In each of the following, find an equation of the circle (or of each circle) determined by the given conditions and make a sketch. (Let C and r represent center and radius.)

   (a) $C = (3,-5)$, $r = 7$
   (b) $C = (-5,12)$ and passing through the origin
   (c) $C = (3,2)$ and tangent to an axis
   (d) $r = 3$ and passing through the points $(-1,1)$, $(2,4)$
   (e) $C = (1,2)$ and tangent to the line $3x - 4y - 12 = 0$
   (f) passing through the points $(2,3)$, $(5,1)$, $(0,1)$

3. (a) Use the fact that a tangent to a circle is perpendicular to the radius at the point of contact to find an equation of a tangent to the circle $x^2 + y^2 = 25$ at the point $(3,-4)$.

   (b) Prove that an equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1,y_1) of the circle is $x_1x + y_1y = r^2$.

4. (a) Find the length of a tangent from $(3,7)$ to the circle $x^2 + y^2 = 25$.

   (b) Show that if t is the length of a tangent from the point (x_1,y_1) to the circle $x^2 + y^2 + Dx + Ey + F = 0$,
   $$t^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F .$$

   (c) If in using this formula you find that $t^2 = 0$, how do you interpret this geometrically? What if $t^2 < 0$?

5. In Section 6-5 we considered the family of circles through the common points of two circles; such a family is sometimes called a coaxial family or a pencil of circles.

   (a) Find an equation of a pencil of circles through the intersections of the circles with equations
   $x^2 + y^2 - 10x - 2y - 35 = 0$ and
   $x^2 + y^2 + 4x - 6y - 49 = 0$.

   (b) Find an equation of a circle of this pencil which passes through the point $(0,-6)$.

   (c) Find an equation of a circle of this pencil which has its center on the line $x + 5 = 0$.

6. In Section 6.5 we found the equation of a line through the common points of two circles; the same algebraic technique gives us the equation of a line, whether the circles intersect or not. This line is called the radical axis of the two circles. Prove that the tangents drawn to two circles from any point in their radical axis are equal in length.

7. Find the coordinates of a point from which equal tangents can be drawn to the three circles with equations $x^2 + y^2 = 4$, $x^2 + y^2 - 6x + y = 12$, $x^2 + y^2 + 4x - 3y = 15$.

8. Prove that the radical axis of two circles is perpendicular to the line of centers of the circles.

9. Two intersecting circles are said to be orthogonal if the tangents at each point of intersection are perpendicular. Prove that if circles $x^2 + y^2 + D_1x + E_1y + F_1 = 0$ and $x^2 + y^2 + D_2x + E_2y + F_2 = 0$ are orthogonal, then $D_1D_2 + E_1E_2 = 2(F_1 + F_2)$.

10. Show that the following pairs of circles are orthogonal.

    (a) $x^2 + y^2 + 3x - 5y + 6 = 0$, $x^2 + y^2 + 10x + 9 = 0$

    (b) $2x^2 + 2y^2 + 2x + 1 = 0$, $2x^2 + 2y^2 - 4x + 6y - 3 = 0$

11. Determine the constant k so that each of the following pairs of circles is orthogonal.

    (a) $x^2 + y^2 - 3x + 4y - 3 = 0$, $x^2 + y^2 + 2x - y + k = 0$

    (b) $3x^2 + 3y^2 + kx - 2y = 4$, $5x^2 + 5y^2 - x + 2y = 2$

Challenge Problems

1. The vertices of triangle ABC are the centers of any three circles which intersect each other. Prove that their common chords are concurrent.

2. The vertices of triangle ABC are the centers of any three circles. Prove that their radical axes are concurrent. (Does your proof also hold for Challenge Problem 1 ?)

7-7. The Ellipse

The ellipse is defined as the set of points P such that the distance from P to a fixed point (the focus) is equal to the product of a constant e and the distance from P to a fixed line (the directrix). The constant e, the eccentricity, is such that $0 < e < 1$. In our earlier study we found that if we take as focus $F = (c,0)$, and as directrix the line $x = \frac{c}{e^2}$, and let $a = \frac{c}{e}$ and $b = \frac{c}{e}\sqrt{1 - e^2}$, an equation for the ellipse can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 . \tag{1}$$

We note from these relations that the equation of the directrix can also be written $x = \frac{a^2}{c}$, or $x = \frac{a}{e}$. Another useful relation is $c^2 = a^2e^2 = a^2 - b^2$.

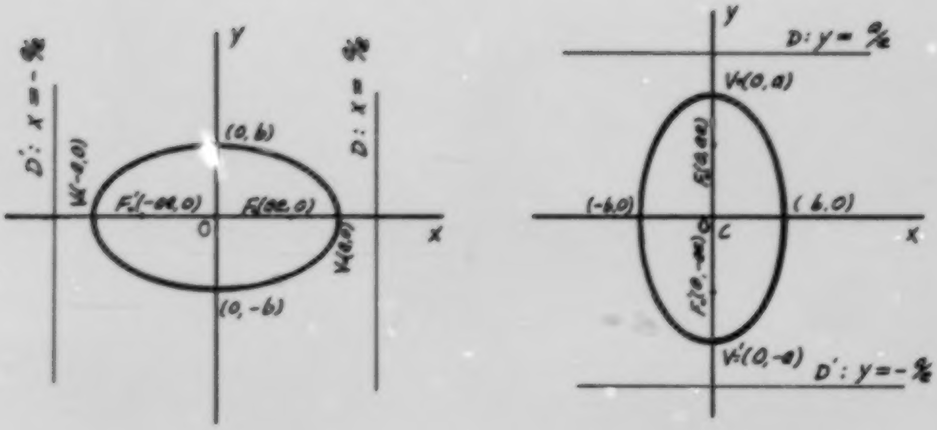
From Equation (1) we see that the graph of the ellipse is symmetric with respect to the origin and to both of the coordinate axes; hence the point $F' = (-c,0)$ and the line $x = -\frac{c}{e^2}$ also serve as focus and directrix. The chord of the ellipse which contains the foci is called the major axis; its endpoints are called vertices. The midpoint of the major axis is called the center of the ellipse; the chord perpendicular to the major axis at the center is called the minor axis.

In Figure 7-4, parts (a) and (c) summarize information about the ellipse with Equation (1), and also the comparable case with the role of the x- and y-axes interchanged.

The equation

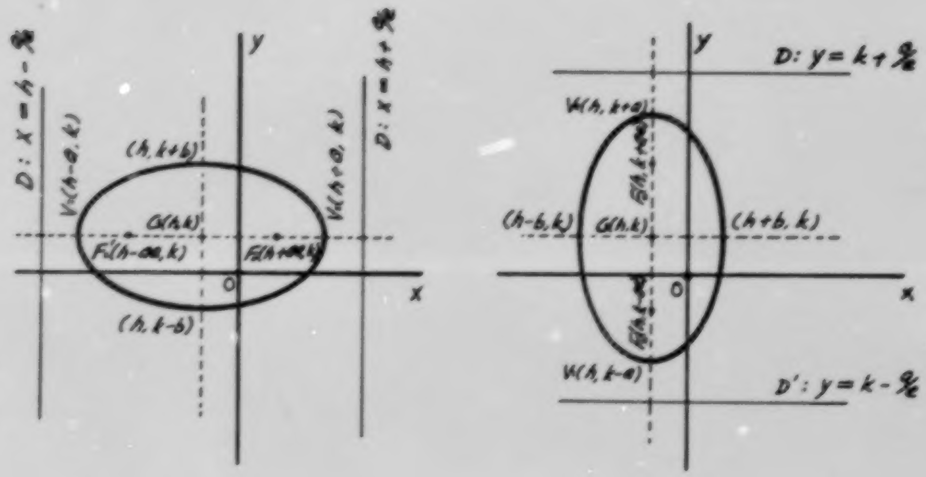
$$\frac{(x - h)^2}{M} + \frac{(y - k)^2}{N} = 1 , \tag{2}$$

if M and N are positive, is in the form of an equation of an ellipse with center $C = (h,k)$. Whether the major axis is parallel to the x- or the y-axis depends on whether M or N is larger. Using V,V', F,F', and D,D' to indicate vertices, foci, and directrices, we can summarize in Figure 7-4, parts (b) and (d), information about an ellipse with center (h,k) and axes parallel to the coordinate axes.


D': x = -a/e
y
D: x = a/e
D: y = a/e
V(0,a)
(0,b)
V'(-a,0)
F'(-ae,0)
F(ae,0)
0
x
V(a,0)
(0,-b)
F(0,ae)
(-b,0)
(b,0)
C
x
F'(0,-ae)
V'(0,-a)
D': y = -a/e

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(c) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$


D: x = h - a/e
y
D: x = h + a/e
D: y = k + a/e
V(h, k+a)
V'(h-a, k)
(h, k+b)
V(h+a, k)
F(h, k+ae)
C(h,k)
F'(h-ae, k)
F(h+ae,k)
0
x
(h, k-b)
(h-b,k)
C(h,k)
(h+b, k)
0
x
F(h, k-ae)
V(h, k-a)
D': y = k - a/e

(b) $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

(d) $\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

For all figures, $a > b$, and $e = \frac{\sqrt{a^2 - b^2}}{a} < 1$

Figure 7-4

If in Equation (2) M and N are negative, there is no locus; sometimes in this case we speak of an imaginary ellipse. The equation

$$\frac{(x - h)^2}{M} + \frac{(y - k)^2}{N} = 0$$

has as its locus only the point (h,k). Such a locus is spoken of as a degenerate ellipse or a point-ellipse, since its equation resembles that of an ellipse.

In discussing the ellipse and its properties and graph we have, in this section, written the equations in rectangular coordinates. All of the work could have been done using polar coordinates. If the equation of an ellipse, or any conic section, is in polar coordinates, you may leave it in that form in order to graph it and obtain such information as coordinates of foci and vertices.

The shape of an ellipse varies with its eccentricity. As you see in Figure 7-5, the nearer e is to zero, the closer the shape of the ellipse is to a circle. You can see why the circle is spoken of as an ellipse of eccentricity zero. For increasingly large values of e, the ellipse is more and more elongated. Can you explain this result from the fact that

$b = \frac{c}{e}\sqrt{1 - e^2} = a\sqrt{1 - e^2}$?

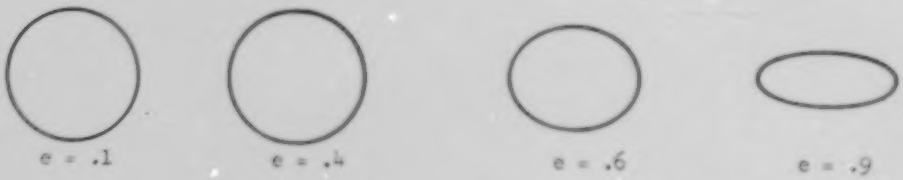


Figure 7-5

Perhaps best known of the properties of an ellipse is that, for any point on an ellipse, the sum of the distances to the foci is a constant equal to the length of the major axis. The reflective property has important applications in optics and radar: Since a tangent at any point of an ellipse makes equal angles with the radii drawn to the two foci, rays are reflected from one focus to the other. This property explains the "whispering gallery" effect in some halls, where a whisper at one spot, though not audible nearby, is easily heard at some more remote spot. The orbits of planets and the paths of electrons about the nucleus in an atom are approximately ellipses with the sun and the nucleus respectively at one focus. The elliptic form also occurs in arches and gears.

Example 1. Discuss and sketch the ellipse with equation

$$9x^2 + 4y^2 + 54x - 16y + 61 = 0 .$$

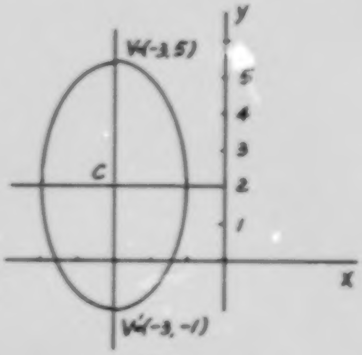
Solution. We proceed to rewrite this equation.

$$9(x^2 + 6x + 9) + 4(y^2 - 4y + 4) = 81 + 16 - 61$$

is equivalent to $$9(x + 3)^2 + 4(y - 2)^2 = 36$$

or $$\frac{(x + 3)^2}{2^2} + \frac{(y - 2)^2}{3^2} = 1 .$$

Since 3 is larger than 2, we see that $a = 3$, $b = 2$, and the major axis is parallel to the y-axis. The curve is an ellipse such as (d) of Figure 7-4 with center $(-3,2)$. The eccentricity $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{5}}{3}$; hence $ae = \sqrt{5}$ and $\frac{a}{e} = \frac{9}{5}\sqrt{5}$. We use these values and the formulas of Figure 7-4 (d) to obtain the coordinates for the vertices, $V = (-3,5)$, $V' = (-3,-1)$, and foci, $F = (-3,2 + \sqrt{5})$, $F' = (-3,2 - \sqrt{5})$, equations of the axes ($x = -3$, $y = 2$) and directrices ($y = 2 \pm \frac{9}{5}\sqrt{5}$). In making a sketch we usually locate the center first, and mark off from it the semi-axes; the values used for this (h,k,a,b) may all be obtained directly from the equation in form (d) of Figure 7-4.



Example 2. Write an equation of the ellipse with foci $F = (2,4)$ and $F' = (-4,4)$ and with $e = \frac{3}{5}$.

Solution. Since for this ellipse the major axis is parallel to the x-axis, we shall use form (b) of Figure 7-4. The distance between the foci is

$$2ae = |2 - (-4)| = 6 ;$$

therefore $$ae = 3 .$$

Since $e = \frac{3}{5}$, $$a = \frac{ae}{e} = \frac{3}{\frac{3}{5}} = 5 .$$

Using the relation $a^2e^2 = a^2 - b^2$,

we have $b^2 = a^2 - a^2e^2$,

$$b^2 = 25 - 9 ,$$

and $b^2 = 16$.

Thus $b = 4$.

Since the center is the midpoint of $\overline{FF'}$, $C = (-1,4)$. We now write the equation

$$\frac{(x + 1)^2}{25} + \frac{(y - 4)^2}{16} = 1 .$$

Exercises 7-7

1. Write an equation of the ellipse with center $(3,2)$, major axis equal to 12 and parallel to the x-axis, and minor axis 8 . Find the eccentricity, the coordinates of the foci and vertices, and the equations of the directrices. Make a sketch.

2. Write an equation of the ellipse with center at $(0,0)$, one vertex $(3,0)$, and one focus $(2,0)$.

3. Rewrite the following equations in the forms of Figure 7-4. For each, find the eccentricity, the coordinates of foci and vertices, and equations of directrices; make a sketch.

   (a) $4x^2 + y^2 = 4$

   (b) $4x^2 + 25y^2 = 100$

   (c) $3x^2 + 2y^2 = 6$

   (d) $4x^2 + 9y^2 = 1$

   (e) $36(x - 4)^2 + 25(y + 3)^2 = 900$

   (f) $4(x + 5)^2 + 9(y + 1)^2 = 36$

   (g) $9x^2 + 4y^2 - 36x = 0$

   (h) $4x^2 + y^2 + 8x - 10y + 13 = 0$

   (i) $16x^2 + 25y^2 - 32x + 150y + 241 = 0$

4. Write an equation of an ellipse to fit each of the following conditions (letters are used as in Figure 7-4).

   (a) $C = (0,0)$; major axis, 8 , parallel to x-axis; minor axis, 6

   (b) $C = (0,0)$; $V = (0,3)$; $F = (0,2)$

   (c) $C = (3,5)$, directrix $x = 10$, $a = 5$

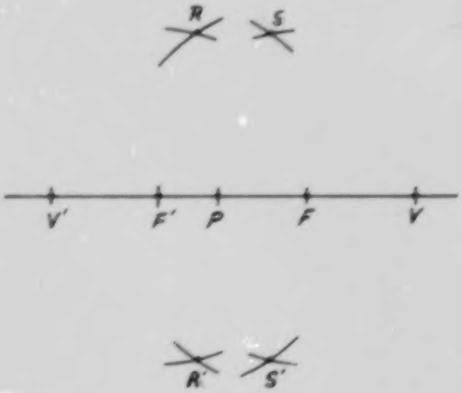
   (d) $F = (3,4)$, $F' = (-1,4)$, $e = \frac{1}{2}$

5. What change must be made in the definition of latus rectum given for the parabola to make it apply to the ellipse? Find a formula for the length of the latus rectum for an ellipse; check that your formula applies for all four cases in Figure 7-4.

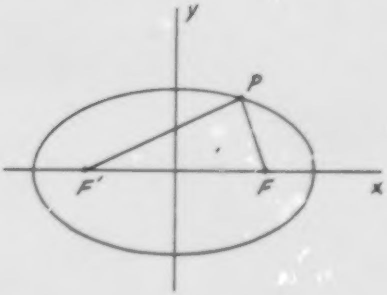
6. A focal radius of an ellipse is a segment drawn from a focus to any point of the ellipse. Prove that the sum of the lengths of the focal radii for any point on an ellipse is a constant, and equal to the length of the major axis.

7. Prove that an ellipse is the locus of points the sum of whose distances from two fixed points is a constant greater than the distance between the two fixed points.

8. Construct some points of an ellipse from given vertices V, V' and foci F, F' as follows. Select any point P of the segment $\overline{V'V}$. With F as center and PV as radius, strike arcs above and below $\overline{V'V}$. With F' as center and PV' as radius, describe arcs intersecting the ones first drawn, and locating points R and R' of the ellipse. Then interchange F and F' and repeat, locating two more points, S and S' . Thus for any point such as P on the segment four points can be located. Why do the points so located lie on the ellipse with the given foci and vertices?



9. Construct an ellipse as follows. Tie the ends of a piece of string to two thumbtacks. Stick the tacks into a piece of cardboard at F and F'. Draw the string taut with a pencil point (P) and trace a curve. Why is the curve an ellipse? Keeping the length of the string the same, change the distance between the tacks and repeat the construction. What do you observe?



10. Use the locus definition in Exercise 7 in deriving equations of

    (a) an ellipse with fixed points $(2,3)$ and $(6,3)$ and sum of focal radii equal to 6.

    (b) an ellipse with fixed points $(1,1)$ and $(3,5)$ and sum of focal radii equal to 6.

11. Some writers like to include the circle as a special case of an ellipse. If a circle with its center at the origin is to be thought of as an ellipse, then $a = b$. What, then, is e? Is this consistent with the focus-directrix definition of a conic?

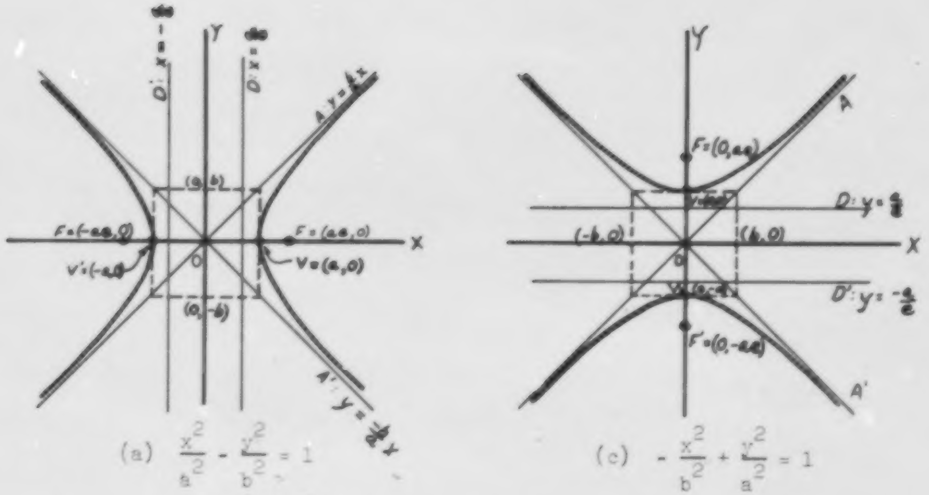
12. Show that the ellipse with focus $F = (c,0)$, eccentricity e, and directrix $x = \frac{c}{e^2}$ has another focus $F' = (-c,0)$ and another directrix $x = -\frac{c}{e^2}$.

13. Discuss and sketch the graph of $r = \frac{6}{2 - \cos \theta}$ including coordinates of the vertices, foci, and center; the lengths of the major and minor axes and of the latus rectum; eccentricity.

14. Prove that in an ellipse the length of the major axis is the mean proportional between the distance between the foci and the distance between the directrices.

7-8. The Hyperbola

The hyperbola is defined as the set of points P such that the distance from P to a fixed point (the focus) is the product of the eccentricity, e, and the distance from P to a fixed line (the directrix), with e greater than one. In our earlier study we found that if, as with the ellipse, we take as focus $F = (c,0)$ and as directrix the line $x = \frac{c}{e^2}$, and let $a = \frac{c}{e}$ and

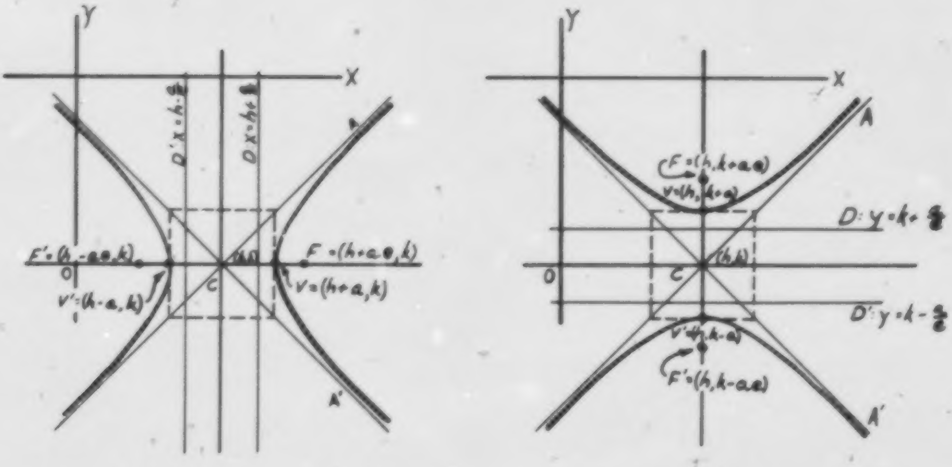

D': x = -a/e
Y
D: x = a/e
A: y = b/a x
(0, b)
F' = (-ae, 0)
F = (ae, 0)
X
V' = (-a, 0)
O
V = (a, 0)
(0, -b)
A': y = -b/a x
Y
A
F = (0, ae)
V = (0, a)
D: y = a/e
(-b, 0)
(b, 0)
X
O
V' (0, -a)
D': y = -a/e
F' = (0, -ae)
A'

(a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$)

(c) $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

(Asymptotes : $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 0$)


Y
X
D': x = h - a/e
D: x = h + a/e
A
F' = (h - ae, k)
O
(h, k)
F = (h + ae, k)
V' = (h - a, k)
C
V = (h + a, k)
A'
Y
X
A
F = (h, k + ae)
V = (h, k + a)
D: y = k + a/e
O
C
(h, k)
D': y = k - a/e
V' = (h, k - a)
F' = (h, k - ae)
A'

(b) $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$

(Asymptotes: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 0$)

(d) $-\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

(Asymptotes : $-\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 0$)

For all figures, $e = \frac{\sqrt{a^2 + b^2}}{a} > 1$

Figure 7-6

$b = \frac{c}{e}\sqrt{e^2 - 1}$, an equation for the hyperbola can be written

(1) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

The hyperbola has the same symmetries as the ellipse. The formulas for foci, vertices, and directrices are also the same; these are summarized for the various simple cases in Figure 7-6.

Unlike the ellipse, the hyperbola is not a bounded curve. In part (a) of Figure 7-6, for example, we see that if we take increasingly large values for x , the corresponding values for y are increasingly large in absolute value. On the other hand, there are values of x (in this case $-a < x < a$) for which there are no real values of y . If we solve (1) for y we get

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} .$$

For very large values of x , the values of y in the first quadrant are very nearly equal to $\frac{b}{a}x$ (corresponding comments apply in the other quadrants). Thus we see intuitively that for values of x that are sufficiently large in absolute value, the distance between a point on the curve and the line with equation $y = \frac{b}{a}x$ (or $y = -\frac{b}{a}x$) can be made arbitrarily small. Thus these lines are asymptotes of the hyperbola; in Figure 7-6 they are marked A and A' . You may wish to refer to Section 6-3 where there is a detailed discussion of the asymptotes of a particular hyperbola; it applies here.

To make a sketch of a hyperbola we first locate the vertices, and then draw the asymptotes. They are drawn easily since they are diagonals of the rectangle with sides $2a$ and $2b$, located as in Figure 7-6. The segment $\overline{VV'}$, of length $2a$, is called the transverse (or major) axis of the hyperbola. (The line segment joining the points $(0,b)$ and $(0,-b)$, of length $2b$, of part (a) of Figure 7-6 is sometimes called the conjugate axis.) From the relationship $c^2 = a^2 + b^2$, we see that the length of the diagonal of the rectangle is also the distance between the foci. We may use this fact to locate the foci.

Conjugate hyperbolas are concentric hyperbolas with the roles of the transverse and conjugate axes interchanged. The equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represent conjugate hyperbolas. As shown in Figure 7-7, they have the same asymptotes, and their foci lie on a circle with center at the center of the curves.

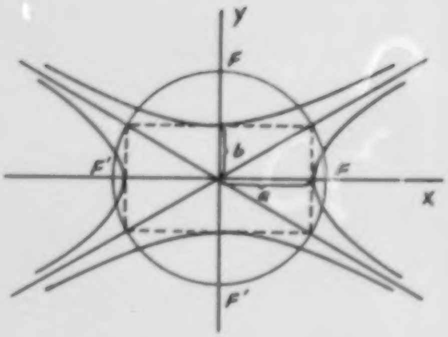


Figure 7-7

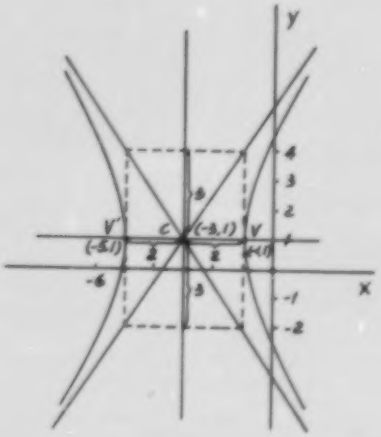
A hyperbola is called equilatera[illegible] rectangular) if the transverse and conjugate axes are equal. In this [illegible] rectangle we have used in sketching is a square, and the asymptotes (which are diagonals) are perpendicular. You may have studied the family of equilateral hyperbolas with equation $xy = k$. These are hyperbolas with the coordinate axes as asymptotes.

For any point of a hyperbola, the absolute value of the difference of its distances from two fixed points is a constant. This property is sometimes used to define a hyperbola; it has applications in range finding and LORAN (Long Range Navigation). Both of these use intersections of families of hyperbolas. As with the ellipse, a tangent at any point of a hyperbola makes equal angles with radii drawn to the foci; for the hyperbola, however, the radii are on opposite sides of the tangent.

Example. Find the equations of the asymptotes of the hyperbola with equation $9x^2 - 4y^2 + 54x + 8y + 41 = 0$. Sketch the curve and its asymptotes.

Solution. We rewrite the equation, following the same procedure as in Example 1 in Section 7-7, getting the equation

$$\frac{(x + 3)^2}{2^2} - \frac{(y - 1)^2}{3^2} = 1 .$$



This is in form (b) of Figure 7-6, with transverse axis having a length of 4, the conjugate axis 6; the center is $C = (-3,1)$. To obtain the equations of the asymptotes, we write

$$\frac{(x + 3)^2}{2^2} - \frac{(y - 1)^2}{3^2} = 0$$

or $3x + 2y + 7 = 0$ and $3x - 2y + 11 = 0$. To make the sketch, we locate the center C, draw through C lines parallel to the coordinate axes, and mark off on them the lengths of the semi-axes. Next we draw the rectangle, its diagonals give the asymptotes, and we can sketch the curve.

Exercises 7-8

1. Write an equation of a hyperbola with semi-axes 2 and 3, center at the origin, and transverse axis on the x-axis. Find the eccentricity, the coordinates of the vertices and foci, and equations of the directrices and asymptotes. Sketch the curve.
2. Repeat Exercise 1, but this time let the transverse axis be on the y-axis.
3. Write an equation of a hyperbola with center $(-2,3)$, semi-axes 4 and 3, and transverse axis parallel to the x-axis. Find the eccentricity, coordinates of vertices and foci, and equations of directrices and asymptotes. Sketch the curve.
4. Repeat Exercise 3, but this time have the transverse axis parallel to the y-axis.

5. For each hyperbola whose equation is given, find the eccentricity and the length of the semi-axes; the coordinates of center, foci, and vertices; the equations of the directrices and asymptotes. Sketch the curves.

   (a) $x^2 - y^2 = 4$

   (b) $y^2 - x^2 = 4$

   (c) $4x^2 - 9y^2 = 36$

   (d) $144y^2 - 25x^2 = 3600$

   (e) $x^2 - 4y^2 - 4x + 24y - 16 = 0$

6. For each part of Exercise 5, write an equation of the conjugate hyperbola.

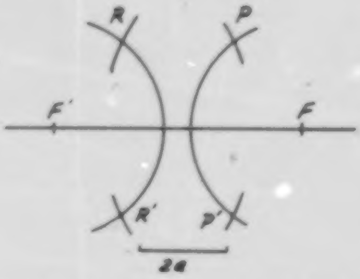
7. Find an equation of the locus of a point such that the absolute value of the difference of its distances from the points $(5,0)$ and $(-5,0)$ is 6.

8. Find an equation of the locus of a point such that the absolute value of the difference of its distances from the points $(1,1)$ and $(-1,-1)$ is 2. What is the eccentricity of this curve?

9. Prove that a hyperbola is the locus of a point such that the absolute value of the difference of its distances from two fixed points is a constant which is less than the distance between the fixed points.

10. What is an appropriate definition of the latus rectum of a hyperbola? Find a formula for the length of the latus rectum of a hyperbola; check that your formula applies in all four cases of Figure 7-6.

11. Construct some points of a hyperbola as follows. Select fixed points F, F' and a length $2a$ $\left(2a < d(F,F')\right)$. With F as center and any desired radius r, describe an arc. With F' as center and radius of length $r + 2a$, describe an arc intersecting the first arc at points P and P'. Then use F' as a center with radius r, and F with radius $r + 2a$, obtaining points R and R'. Thus for a particular choice of r, four points can be located. Why do the points so located lie on a hyperbola?



12. Prove that the equations $x = a \sec \theta$, $y = b \tan \theta$ are a parametric representation of a hyperbola.

13. See if you can devise a method of constructing a hyperbola which uses the equations in Exercise 12. (Hint: See Section 5-4.)

14. Find equations of the equilateral hyperbolas through the point $(3,-7)$,

    (a) with the coordinate axes as asymptotes.

    (b) with axes of the hyperbola along the coordinate axes.

15. Just as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ was considered an equation of a degenerate ellipse, we may speak of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ as the equation of a degenerate hyperbola. What is the locus in this case?

7-9. Summary

A conic section is the intersection of a plane and a right circular cone; it is a circle, ellipse, parabola, hyperbola, or, in a degenerate case, a point, line, or pair of lines.

In polar coordinates a circle with center at the origin has the equation $r = k$. Any other conic section may be defined as the locus of points in a plane such that for each point the ratio of its distance from a given point in the plane to its distance from a given line in the plane is a constant e, called the eccentricity. Such a conic, if the center is at the pole and directrix perpendicular to the polar axis and p units to the left of it, has the equation

$$r = \frac{ep}{1 - e \cos \theta}, \qquad \text{representing} \quad \begin{cases} \text{a parabola if } e = 1, \\ \text{an ellipse if } 0 < e < 1, \\ \text{a hyperbola if } e > 1. \end{cases}$$

The equations that relate polar and rectangular coordinates were used to find corresponding rectangular equations. These were seen to be equivalent to the equations developed in earlier work in algebra. Since the information about the conics in rectangular form is summarized at the beginning of the sections (7-5 through 7-8) dealing with each type, it is not repeated again here.

Conic sections have wide usefulness in theoretical work in mathematics and science, and in applications to a great variety of problems in science and industry; it has been possible to mention only a few here.

With this chapter we conclude, for the time being, our study of the analytic geometry of two-space. We shall take up next the analytic geometry of three-space. Later, if time permits, there may be an opportunity to return again to conic sections in order to consider the general problem of showing that all equations of second degree in x and y have loci which are conic sections, and then to relate the corresponding algebraic and geometric properties.

Review Exercises

1. Sketch the graph of each of the following equations. Identify each conic section, and give the appropriate information (foci, vertices, center, eccentricity, directrices, asymptotes, etc.) .

   (a) $3r - 2 = 0$

   (b) $r = 2 \cos \theta$

   (c) $r = \dfrac{8}{1 - \cos \theta}$

   (d) $r = \dfrac{4}{2 - 3 \cos \theta}$

   (e) $2 - \cos \theta = \dfrac{3}{r}$

   (f) $r = \dfrac{12}{3 - 3 \cos \theta}$

   (g) $4r = 3r \cos \theta + 24$

   (h) $r = 4 - r \sin \theta$

   (i) $r = 3 + 2r \cos \theta$

   (j) $x^2 - 4x + y^2 + 6y + 13 = 0$

   (k) $3x^2 - 2y^2 = 6$

   (l) $y^2 + 8x - 6y + 25 = 0$

   (m) $25x^2 + 36y^2 + 100x + 288y - 224 = 0$

   (n) $3x^2 + 5y^2 - 6x + 20y + 8 = 0$

   (o) $x^2 + y^2 - 6x + 10y + 34 = 0$

   (p) $x^2 - 3y^2 + 8x - 6y - 14 = 0$

   (q) $144x^2 - 25y^2 + 576x + 150y - 3249 = 0$

2. Write an equation for each of the following and sketch the graph.

   (a) A parabola with vertex $(0,0)$ and focus $(-5,0)$.

   (b) A parabola with vertex $(7,6)$ and directrix $y = -2$.

   (c) A circle with radius 5 and tangent to both axes.

   (d) A circle with center $C = (1,-4)$ and passing through $(3,-2)$.

   (e) A circle tangent to the line $x - 2y - 2 = 0$, passing through the point $(-2,0)$, and with center on the y-axis.

   (f) A circle passing through the points $(0,4)$, $(6,6)$ and $(-2,-10)$.

   (g) An ellipse with center $(2,3)$, a vertex $(5,3)$, and a directrix $x = -4$.

   (h) An ellipse with a focus $(-3,5)$, and directrices $y = 6$ and the x-axis.

   (i) A hyperbola with foci $(-1,1)$ and $(5,1)$, and a vertex $(0,1)$.

   (j) A hyperbola with asymptotes $3x - 4y = 0$, $3x + 4y = 0$, and passing through the point $(3,5)$.

   (k) A parabola with axis parallel to the y-axis, passing through the points $(2,11)$, $(0,5)$ and $(-1,8)$.

3. Find an equation of the locus of a point whose distance from the point $(-1,4)$ is 2 units more than its distance from the line $y + 2 = 0$.

4. Find an equation of the locus of the center of a circle which is tangent to the line $x = 3$ and passes through $(1,-1)$. Explain from geometric considerations why this locus must be a parabola.

5. Find the eccentricity of an ellipse whose major axis is twice the length of its minor axis.

6. Prove that the equations $x = a \cos \theta$, $y = b \sin \theta$ are a parametric representation of an ellipse.

7. Find an equation of the locus of a point which moves so that its distance from the point $(0,2)$ is one-half its distance from the point $(3,1)$.

8. Prove that the product of the distances from any point on a hyperbola to the asymptotes is a constant.

9. (a) If the ratio of the length of the conjugate axis to the length of the transverse axis of a hyperbola is 2, what is the eccentricity?

   (b) If the ratio is k, find a formula for e.

10. (a) Show that $x = \dfrac{tr}{\pm\sqrt{1 + t^2}}$, $y = \dfrac{r}{\pm\sqrt{1 + t^2}}$ are parametric equations of a circle. (These equations are sometimes useful in calculus.)

(b) What is the graph of the equations in (a) if only the positive signs before the radicals are used? If only the negative signs?

(c) Show that these parametric equations do not represent the points $(r,0)$ and $(-r,0)$. Since this is the case, what would be a more precise way to state (a) in this exercise?

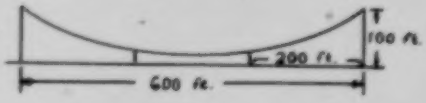
11. Prove that, for the conjugate hyperbolas whose equations are

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the sum of the squares of the reciprocals of the eccentricities is one.

12. A curve is defined by the parametric equations $x = a + k \cos \theta$, $y = b + k \sin \theta$, where a , b , and k are arbitrary constants $(k \neq 0)$. Find an equation of the curve in standard rectangular form and identify it. What is the significance of the requirement that k not be zero?

13. An archway is in the shape of a semi-ellipse. The distance across the base of the archway is 30 feet, and its maximum height from the base is 20 feet. What should be the limit on the height of vehicles using a centrally placed 20-foot wide road under the archway? (The posted limit is such that a vehicle of that height, at the edge of the road but not off the road, will have clearance.)

14. The cable of a suspension bridge hangs in the form of a parabola from supporting towers 600 feet apart. The points where the cable is suspended from the towers are 100 feet above the roadway, and the lowest part of the cable is 10 feet above the roadway. If there are supporting structures to the cable from the two points on the roadway each 200 feet from the base of the towers, how high must these supporting structures be?



15. Prove that the product of the focal radii from a point on an equilateral hyperbola is equal to the square of the distance from the point to the center.

16. (a) Write an equation of the family of ellipses with the origin as center, major axis along the x-axis, and eccentricity equal to $\frac{3}{5}$.

    (b) Write an equation for the member of this family with the length of the minor axis equal to 12.

    (c) Write an equation for the member of this family which passes through the point $(4,\frac{12}{5})$.

17. Prove the following statements analytically.

    (a) A radius perpendicular to a chord bisects the chord.

    (b) The perpendicular from any point of a circle to a diameter is the mean proportional between the segments of the diameter.

    (c) The locus of a point such that its distance from one fixed point is a constant multiple of its distance from a second fixed point is a circle. (What restriction must there be on the value of the constant for this to be a correct statement?)

Challenge Problems

1. Prove that in a hyperbola an asymptote, a directrix, and a line from the corresponding focus perpendicular to the asymptote are concurrent.

2. On a map marked with a rectangular grid using a mile as a unit, three listening posts are at $A = (0,0)$, $B = (2,0)$, and $C = (0,4)$. An explosion is heard at A 5 seconds after it is heard at B, and 8 seconds after it is heard at C. Where did the explosion take place? (Use 0.2 mile per second as the speed of sound. Find equations of the two loci involved, and find the appropriate intersection either by graphing or by using the equations of the asymptotes. Do you think that it is sufficiently accurate in this case to assume that the asymptotes meet at the point you want?)

3. A taxpayer changes his residence because of a change in his place of work. For his moving expenses to be allowed as a deduction under the Revenue Act of 1964, it is necessary (among other requirements) that his new principal place of work be "at least 20 miles farther from his former residence than was his former principal place of work."

   Suppose a man's new employment is at a place 30 miles from where he was previously employed. Let $P = (x,y)$ represent the location of his old home. Write in analytic form the condition under which the man would be entitled to deduct moving expenses to a new home. (Suggestion: If W_1 and W_2 are points representing the old and new places of employment respectively, let $\overleftrightarrow{W_1W_2}$ be the x-axis, and let the midpoint of $\overline{W_1W_2}$ be the origin.)

4. For the parabola $r = \frac{6}{1 - \cos\theta}$, prove the reflective property, that is, the tangent to the parabola at the point $P = (r,\theta)$ makes equal angles with the polar radius $\overline{OP}$ and the line through P parallel to the polar axis.

5. Prove analytically that, in any triangle, the midpoints of the sides, the feet of the altitudes, and the points halfway between the vertices and the orthocenter lie on a circle. This is called the *nine-point* circle.

Chapter 8

THE LINE AND THE PLANE IN 3-SPACE

8-1. The Extension to 3-Space.

To this point in our study we have sought analytic representations of subsets of a plane; in turn we have sketched the loci, or graphs, of both algebraic and vector relationships with the assumption, usually tacit, that their geometric interpretation was confined to a plane or a line.

Our previous experience in geometry has been largely in a plane; even when we did consider geometric configurations in space, we frequently pursued our investigations in only one or two planes.

It is easier to analyze loci in a plane, but we live in a world of three dimensions. If we are to apply our geometric knowledge to physical problems, we must be able to extend our concepts to 3-space.

In this chapter and the next we shall consider the basic extension to 3-space of the ideas which we have already developed; we shall even suggest how repetition of this process leads to mathematical structures with more dimensions, which are called spaces, even though we cannot possibly visualize them.

In this chapter we shall be extending some of the ideas of Chapters 2 and 3 to 3-space; you might want to review these chapters briefly before you continue. We assume that you have had some experience with rectangular coordinate systems in 3-space, but we shall reconstruct the development. We shall consider the analytic representations of lines and planes, and we shall make suggestions on sketching to help you visualize their graphs. The extension of vectors to spaces of higher dimension is surprisingly easy; this is another reason for the favor vectors find in contemporary analysis.

One thing you might keep in mind. The locus of a condition depends upon the space to which it is applied. We have already seen that the equation $x = 1$ describes both a point on a line and a line in a plane. Here we shall see that it also describes a plane in 3-space. In spaces of higher dimension it would be subject to still other interpretations. In general, analytic representations describe loci in any space which has at least as many dimensions as the analytic representation has independent variables. To describe the locus we must first know the number of dimensions of the space in which it occurs.

8-2. *A Coordinate System for 3-Space.*

In Sections 2-1 and 2-3 we discussed rectangular coordinate systems on a line and in a plane. Now we shall indicate how a similar coordinate system can be introduced into 3-space.

We begin by selecting an arbitrary point O in space and three mutually perpendicular lines through O. The point O is called the *origin* of the coordinate system and the lines are called the *x-*, *y-*, and *z-axes*. On each axis we set up a linear coordinate system with point O as its origin. The plane determined by the x- and y-axes is called the *xy-plane*. The *xz-* and *yz-planes* are defined similarly. The three are called the *coordinate planes*. Let P be any point in space. Let a be the coordinate of the projection of P on the x-axis. a is called the *x-coordinate* of P. The *y-* and *z-coordinates*, say b and c respectively, are defined similarly. To the point P we assign the *ordered triple* (a,b,c) of coordinates. Just as in the plane, the correspondence between points and ordered sets of coordinates is one-to-one. The coordinate planes divide space into eight regions called, not unnaturally, *octants*. Usually only one of them is numbered, and it is called the first and is the one in which all the coordinates of every point are positive.

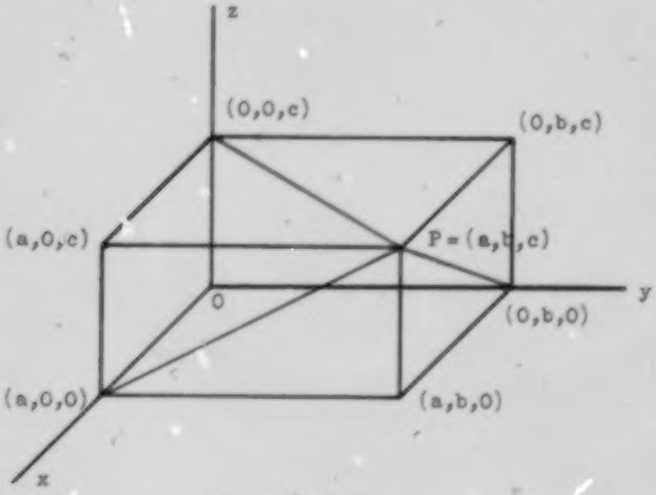


Figure 8-1

The point $(a,b,0)$ is called the projection of (a,b,c) on the xy-plane. The point $(a,0,0)$ is called the projection of (a,b,c) on the x-axis, and so forth.

The configuration of axes shown in Figure 8-1 is called a right-handed system because a 90° rotation of the positive side of the x-axis into the positive side of the y-axis will advance a right-handed screw along the positive side of the z-axis. We shall use this system in drawings in this text. If the locations of the x- and y-axes are interchanged, as you will find that they are in some texts, the system is left-handed.

Distance Between Two Points: We may use the Pythagorean Theorem to develop a formula for the distance between two points in space. If the points are $P_0 = (x_0,y_0,z_0)$ and $P_1 = (x_1,y_1,z_1)$, the distance between them is

$$d(P_0,P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}\,. \tag{1}$$

Points of Division: An extension to 3-space of the method used in Section 2-3 to obtain the coordinates of the point which divides a line segment in the ratio $\frac{c}{d}$ gives us, for the segment P_0P_1,

$$x = \frac{dx_0 + cx_1}{c + d}$$
$$y = \frac{dy_0 + cy_1}{c + d} \tag{2}$$
$$z = \frac{dz_0 + cz_1}{c + d}$$

In the special case when $c = d$, we have the midpoint, with

$$x = \frac{x_0 + x_1}{2}$$
$$y = \frac{y_0 + y_1}{2} \tag{3}$$
$$z = \frac{z_0 + z_1}{2}$$

Exercises 8-2

1. Draw a sketch showing each of the following points in space:

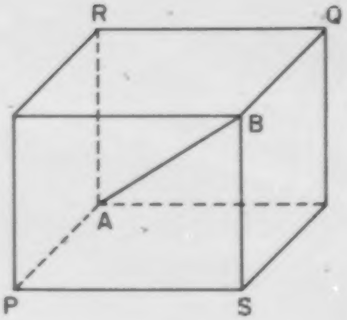
   (a) (1,2,1)
   (b) (-2,1,1)
   (c) (2,0,-1)
   (d) (1,-1,2)
   (e) (-1,-1,2)
   (f) (-1,-2,-1)
   (g) (-3,1,-1)
   (h) (1,-1,-2)

In Exercises 2, and 3, $P = (1,2,3)$, $Q = (-3,2,1)$, and $R = (2,-3,1)$.

2. Find $d(O,P)$, $d(O,Q)$, $d(P,R)$, and $d(Q,R)$.

3. Find the midpoints of $\overline{OP}$ and $\overline{PR}$.

4. (a) Draw $\overline{AB}$, about 3 inches long oblique to the edge of your paper. Consider $\overline{AB}$ as drawn from the rear lower left to the front upper right corner of a rectangular solid. Next draw oblique segments from A to P and from B to Q equal in length and parallel but with opposite sense of direction. If, as is usually the case, the solid is to be oriented with respect to rectangular coordinate axes, make $\overline{AP}$ and $\overline{BQ}$ parallel to the x-axis. Then draw a rectangle with horizontal and vertical sides and with P and B as opposite vertices; this is the front face. The back face is another rectangle with A and Q as opposite vertices. Two more segments complete the figure.



   (b) Now start again with the same kind of diagonal segment $\overline{AB}$, but consider it drawn from the front lower left to the rear upper right, and draw the new solid. This time reverse the directions of $\overline{AP}$ and $\overline{BQ}$. Now A and Q are in the front face and B and P are in the back face.

5. The origin and the point $P = (3,5,4)$ are the opposite corners of a rectangular box that has three of its edges along the axes. Draw the box and give the coordinates of its other vertices.

6. Repeat Exercise 5, using $P = (-5,4,-3)$.

7. Given: $P_1 = (2,-3,4)$ and $P_2 = (-1,3,-2)$

   (a) Make a drawing which shows P_1, P_2, and $\overline{P_1P_2}$.

   (b) Write the coordinates of the points which are the projections of P_1 and P_2 on each of the axes and on each of the coordinate planes.

   (c) Find the length of $\overline{P_1P_2}$ and the length of its projections on the axes and on the coordinate planes.

8. Repeat Exercise 7, using $P_1 = (-3,5,7)$ and $P_2 = (3,0,-3)$.

9. If $P_1 = (3,-4,6)$ and $P_2 = (-2,3,-2)$ find the coordinates of point P on $\overleftrightarrow{P_1P_2}$ if

   (a) P is the midpoint of $\overline{P_1P_2}$.

   (b) $d(P_1,P) = \frac{1}{2}d(P,P_2)$

   (c) $d(P_1,P) = \frac{3}{5}d(P,P_2)$

   (d) $d(P_1,P) = \frac{5}{3}d(P,P_2)$

   (e) $d(P_1,P) = \frac{3}{5}d(P_1,P_2)$

   (f) $d(P_1,P) = \frac{5}{3}d(P_1,P_2)$

10. In triangle ABC, $A = (2,4,1)$, $B = (1,2,-2)$ and $C = (5,0,-2)$. Find the lengths of the sides of this triangle and decide what kind of triangle it is.

Challenge Problem

We introduced a coordinate system in 3-space by selecting three mutually perpendicular lines through an arbitrary point. Show that this is possible.

8-3. Parametric Representation of the Line in 3-Space.

Our discussion in Section 5-6 of the parametric representation of a line in a plane generalizes quite easily to 3-space. Let $P_0(x_0,y_0,z_0)$ and $P_1(x_1,y_1,z_1)$ be two points in space and let L be the line through them.

Assume for the time being that L is not parallel to or lying in any coordinate plane. Then P_0 and P_1 cannot both lie in the xy-plane and we let P_1 be one which does not. Hence P_0, P_1, and $(x_1, y_1, 0)$ are not collinear and determine a plane M containing L. M intersects the xy-plane in a line L' called the projection of L on the xy-plane. Since the line containing P_1 and $(x_1, y_1, 0)$ is perpendicular to the xy-plane, plane M is perpendicular to the xy-plane. Hence the line from P_0 perpendicular to the xy-plane (and thus intersecting it in the point $(x_0, y_0, 0)$) lies in plane M and is a point of L, the line of intersection.

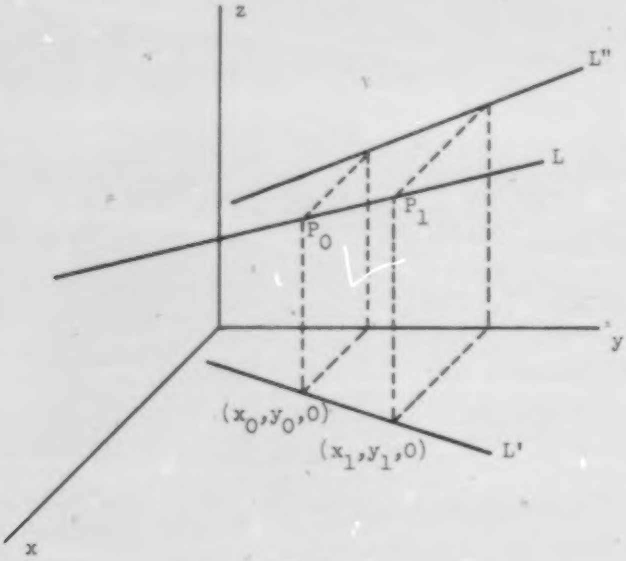


Figure 8-2

From our previous discussion, we know that L' has the parametric equations

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \, . \end{aligned} \tag{1}$$

We would have a parametric representation for L very similar to the one we obtained for a line in a plane if we could show that if $P = (x, y, z)$ is on L

$$z = z_0 + t(z_1 - z_0) \, .$$

Clearly

$$z = z_0 + s(z_1 - z_0)$$

for suitable s. The question is, is s equal to t? That it is can be proved as follows. Let L'' be the projection of L on the yz-plane. Then in this plane L'' has the parametric representation

$$(2) \quad \begin{aligned} y &= y_0 + s(y_1 - y_0) \\ z &= z_0 + s(z_1 - z_0) . \end{aligned}$$

From (1) and (2) it follows that for each point $P = (x,y,z)$ of L, $s = t$, and hence L has the parametric representation

$$(3) \quad \begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \\ z &= z_0 + t(z_1 - z_0) . \end{aligned}$$

We leave it to the student as an exercise to prove that (3) represents L even if L is in or parallel to a coordinate plane.

To save writing, let $\ell = x_1 - x_0$, $m = y_1 - y_0$, and $n = z_1 - z_0$. We call (ℓ,m,n) an ordered triple of direction numbers for L. If $c \neq 0$ the equations

$$\begin{aligned} x &= x_0 + c\ell t \\ y &= y_0 + cmt \\ z &= z_0 + cnt \end{aligned}$$

also represent L. Thus it is natural to extend the definition of equivalence of ordered pairs of direction numbers for a line in a plane to ordered triples of direction numbers for a line in space. Two such ordered triples are said to be equivalent if corresponding numbers are proportional.

Let L and L' be the lines with parametric equations

$$L: \begin{cases} x = x_0 + \ell t \\ y = y_0 + mt \\ z = z_0 + nt \end{cases} \qquad L': \begin{cases} x = \ell t \\ y = mt \\ z = nt \end{cases}$$

and assume L does not go through the origin. Then, as we shall prove, L and L' are

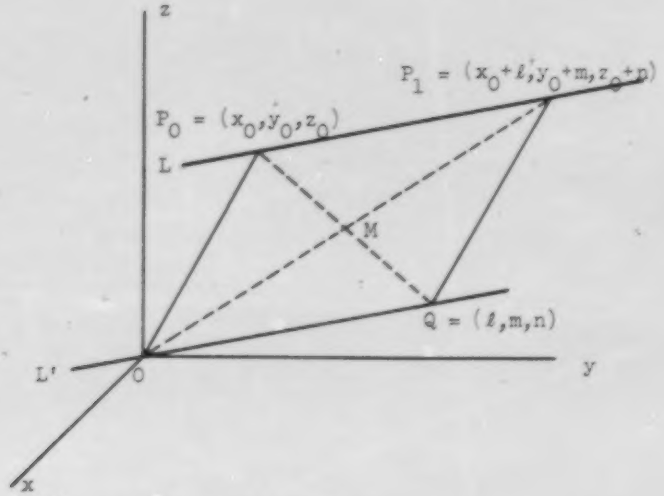


Figure 8-3

parallel. Let $P_1 = (x_0 + \ell,\ y_0 + m,\ z_0 + n)$, $Q = (\ell, m, n)$. Then P_1 and $P_0(x_0, y_0, z_0)$ are on L ; 0 and Q are on L' . The midpoint of $\overline{OP_1}$ is

$$M = \left(\frac{x_0 + \ell}{2},\ \frac{y_0 + m}{2},\ \frac{z_0 + n}{2}\right),$$

M is also the midpoint of $\overline{P_0Q}$. Thus OP_0P_1Q is a plane quadrilateral whose diagonals bisect each other and hence is a parallelogram. It follows that L and L' are parallel. The following theorem is an almost immediate consequence of our argument.

THEOREM 8-1. Two distinct lines L and L' are parallel if and only if any triple of direction numbers for L is equivalent to any one for L' .

As in the plane, a set of direction numbers for a line can be used to establish a direction on the line. Let (ℓ, m, n) be a triple of direction numbers for the line L . If $P_0 = (x_0, y_0, z_0)$ is a point on L , L has the representation

$$x = x_0 + \ell t$$
$$y = y_0 + mt$$
$$z = z_0 + nt\ .$$

The positive ray (on L) with endpoint P_0 is the set of points consisting of P_0 and all points of L given by positive values of t . If P_1 is another point of L , the positive ray with endpoint P_1 points in the same direction as the one with endpoint P_0 in the sense that their intersection is one of them. If $c > 0$, the triple $(c\ell,cm,cn)$ of direction numbers for L establishes the same positive direction on L as does the triple (ℓ,m,n) .

If (ℓ,m,n) is a triple of direction numbers for L , the triple

$$(\lambda,\mu,\nu) = \left(\frac{\ell}{\sqrt{\ell^2 + m^2 + n^2}}, \frac{m}{\sqrt{\ell^2 + m^2 + n^2}}, \frac{n}{\sqrt{\ell^2 + m^2 + n^2}}\right)$$

is of particular importance. Such a triple is sometimes called a normalized triple. Note that $\lambda^2 + \mu^2 + \nu^2 = 1$. Let us assume first that L goes through the origin. The point $P = (\lambda,\mu,\nu)$ lies on L and $d(0,P) = 1$. Figures 8-4a and 8-4b show the situation when $\lambda > 0$, $\mu > 0$, $\nu > 0$ and the situation when $\lambda < 0$, $\mu > 0$, $\nu > 0$ respectively. In both cases, $\mu = \cos \beta$, where β is the angle determined by the positive ray on L with endpoint O and the positive half of the y-axis. α and γ are defined similarly, with the positive halves of the x- and z-axes, respectively, replacing the positive half of the y-axis.

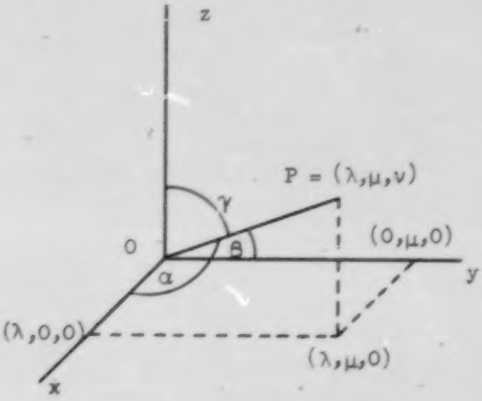


Figure 8-4a

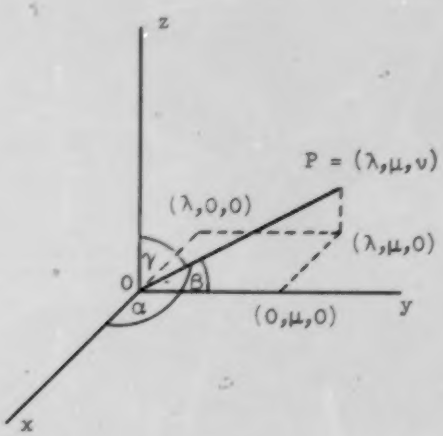


Figure 8-4b

If L is the x-axis, then any triple of direction numbers for it has the form $(\ell,0,0)$. If $\ell > 0$, the positive ray with endpoint O is the positive half of the x-axis and $\cos \alpha = 1$. If $\ell < 0$, the positive ray on L with endpoint O is the negative half of the x-axis and $\cos \alpha = -1$. Similarly, if L is the y-axis, $\cos \beta = \pm 1$ depending on the algebraic sign of m, and if L is the z-axis, $\cos \gamma = \pm 1$ depending on the algebraic sign of n. The student should consider the other possible combinations of signs for λ, μ, and ν, to make sure that in every case $\lambda = \cos \alpha$, $\mu = \cos \beta$, and $\nu = \cos \gamma$. The angles α, β, and γ are called *direction angles* of the line L with its direction determined by the ordered triple (ℓ,m,n) of direction numbers. Their cosines are called the *direction cosines*. If we determine the direction of L by means of the triple $(c\ell,cm,cn)$ of direction numbers, with $c < 0$, and if α', β', and γ' are the new direction angles, then α and α' are supplementary angles, as are β and β', and γ and γ'.

Finally, let L be a line which does not pass through the origin, and let (ℓ,m,n) be an ordered triple of direction numbers for L. Let L' be the line through the origin parallel to L, and let the direction on L' be determined by the triple (ℓ,m,n) of direction numbers. Then we define the direction angles and cosines of L to be the corresponding ones for L'.

Notice that throughout this discussion we do not define direction angles or direction cosines for a line, but only for a line which has been assigned a direction by means of a triple of direction numbers.

In Section 2-3 we derived a parametric representation of points on a line from their symmetric representation. Something similar can be done with a parametric representation of a line in space. Let L be the line with parametric equations

$$\begin{aligned} x &= x_0 + \ell t \\ y &= y_0 + mt \\ z &= z_0 + nt \,. \end{aligned} \tag{4}$$

Suppose that $\ell mn \neq 0$. Then we can eliminate t from any two of these equations by solving each one for t and setting the results equal to each other. Using the first two, we get

$$t = \frac{x - x_0}{\ell} = \frac{y - y_0}{m} \,.$$

Using the first and third, we get

$$t = \frac{x - x_0}{\ell} = \frac{z - z_0}{n} .$$

Combining the last two results we get

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m} = \frac{z - z_0}{n} . \tag{5}$$

These are called symmetric equations for L .

There remains the question of what we have achieved by eliminating t . Let t_0 be any real number and let

$$\begin{aligned} a &= x_0 + \ell t_0 \\ b &= y_0 + m t_0 \\ c &= z_0 + n t_0 . \end{aligned}$$

Then

$$\frac{a - x_0}{\ell} = \frac{b - y_0}{m} = \frac{c - z_0}{n} .$$

Thus if the point (a,b,c) is on the graph of (4) it is also on the graph of (5). If we let

$$t_0 = \frac{a - x_0}{\ell} = \frac{b - y_0}{m} = \frac{c - z_0}{n} ,$$

we find at once that the point (a,b,c) also lies on the graph of (4). Thus the graphs of (4) and (5) are identical.

Equations (5) are equivalent to any pair of the three equations

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m}$$

$$\frac{x - x_0}{\ell} = \frac{z - z_0}{n}$$

$$\frac{y - y_0}{m} = \frac{z - z_0}{n} .$$

Each of these is an equation of a plane. We shall discuss the significance of this particular set of three planes containing a line in the next section.

If at least one of the direction numbers for L vanishes we cannot write such symmetric equations for L . We can, however, eliminate t and obtain equations of two planes containing L . We leave this to the exercises.

You may have read of spaces of four or more dimensions. We are now in a position to give you some idea of what was meant. You have learned how to set up a one-to-one correspondence between the points in a plane and the ordered pairs of real numbers, and between the points in 3-space and the ordered triples of real numbers. Given a coordinate system, it is natural to speak of "the point $(2,3)$ " or "the point $(3,2,-1)$." This suggests that we should define a point in 4-space, for example, to be an ordered quadruple of real numbers. Similarly, we define a line in 4-space to be the set of points in 4-space given by a set of parametric equations of the form

$$\begin{aligned} x &= x_0 + \ell t \\ y &= y_0 + mt \\ z &= z_0 + nt \\ w &= w_0 + pt . \end{aligned}$$

It can then be proved that there is one and only one "line" through two distinct "points." We can define the distance between $P_0(x_0,y_0,z_0,w_0)$ and $P_1(x_1,y_1,z_1,w_1)$ to be

$$d(P_0,P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 + (w_1 - w_0)^2} .$$

We can define the coordinate axes to be the four "lines" through $(0,0,0,0)$ each of which passes through one of the "points" $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. Many other geometric concepts you have studied can be generalized in this way, but that is beyond the scope of this course.

Example. If $A = (3,-1,4)$, $B = (-2,2,1)$ and $C = (2,3,-2)$,

(a) write parametric and symmetric representations for $\overleftrightarrow{AB}$, and

(b) write equations for the line through C parallel to $\overleftrightarrow{AB}$.

Solution.

(a) For parametric form (Equations (4)), we need a point on the line and direction numbers. We choose $A = (3,-1,4)$, and obtain direction numbers $(5,-3,3)$. Hence the line $\overleftrightarrow{AB}$ has as a parametric representation

$$\begin{aligned} x &= 3 + 5t \\ y &= -1 - 3t \\ z &= 4 + 3t \,. \end{aligned}$$

From the first two of these we get

$$t = \frac{x - 3}{5} = \frac{y + 1}{-3} \,.$$

From the last two we get

$$t = \frac{y + 1}{-3} = \frac{z - 4}{3} \,.$$

Combining the last two results, we have as symmetric equations for $\overleftrightarrow{AB}$

$$\frac{x - 3}{5} = \frac{y + 1}{-3} = \frac{z - 4}{3}$$

(b) Since we have direction numbers for $\overleftrightarrow{AB}$, we can write immediately a parametric representation of a parallel line through C ,

$$\begin{aligned} x &= 2 + 5t \\ y &= 3 - 3t \\ z &= -2 + 3t \end{aligned}$$

Exercises 8-3

In Exercises 1 to 3, $P = (1,2,3)$, $Q = (-3,-2,1)$, and $R = (2,-3,1)$.

1. Write parametric equations for the lines determined by the following conditions:

   (a) Through P , parallel to the x-axis
   (b) Through Q , parallel to the z-axis
   (c) Through P and Q
   (d) Through Q and R
   (e) Through O parallel to $\overleftrightarrow{PQ}$
   (f) Through O parallel to $\overleftrightarrow{QR}$
   (g) Through O and P
   (h) Through P , parallel to the xy-plane, and intersecting the z-axis
   (i) Through P parallel to $\overleftrightarrow{QR}$
   (j) Through R parallel to $\overleftrightarrow{PQ}$

2. Write an equation in symmetric form for each of the lines referred to in Exercise 1 (if it is possible to do so).

3. Write a set of normalized direction numbers for each of the lines described in Exercise 1.

4. Find two parametric representations of the line through each of the following pairs of points which establish opposite directions on the line. Find the coordinates of another point on each line.

   (a) $(1,1,-2)$ and $(0,-1,-1)$
   (b) $(-1,-1,-1)$ and $(-2,-1,1)$
   (c) $(4,2,1)$ and $(1,-2,4)$
   (d) $(-3,1,1)$ and $(1,2,-1)$

5. Find the two triples of direction cosines for each line in Exercise 1. Using a table of the values of the trigonometric functions, find the approximate value of each of the direction angles.

6. What are direction cosines for the axes?

7. Find direction cosines of a line that makes equal angles with the axes.

8. In each of the following parts determine whether the third point is on the line containing the first two.

   (a) $(1,1,-2)$, $(0,-1,-1)$, $(2,3,-2)$
   (b) $(1,0,1)$, $(-1,-1,-2)$, $(-7,-4,-11)$

9. Determine which, if any, of the lines determined by the following pairs of points are parallel.

(a) (1,1,-2) and (-1,2,3)
(b) (3,-1,2) and (-1,1,11)
(c) (1,-1,3) and (5,1,11)
(d) (-3,5,12) and (1,3,3)
(e) (2,-3,4) and (-2,-5,-6)
(f) (-1,0,1) and (1,-1,-4)

10. Write symmetric equations for the lines

$$L_1: \begin{cases} x = 2 + 3t \\ y = 1 - 2t \\ z = -1 - t \end{cases} \qquad L_2: \begin{cases} x = -1 + t \\ y = 2 + 2t \\ z = 4 - t \end{cases}$$

$$L_3: \begin{cases} x = 3 + 2t \\ y = -5 - 3t \\ z = 4t \end{cases} \qquad L_4: \begin{cases} x = 2 - t \\ y = -1 + 3t \\ z = -2 \end{cases}$$

11. Prove that if L has the parametric representation $x = x_0 + \ell t$, $y = y_0 + mt$, $z = z_0 + nt$, and if P_1 and P_2 are the points on L given by the values $t = t_1$ and $t = t_2$, then

$$d(P_1P_2) = \sqrt{\ell^2 + m^2 + n^2}\,|t_2 - t_1| .$$

Interpret this result in words, including the special case when the direction numbers are normalized.

12. Prove that Equations (3) represent L even if L is in or parallel to a coordinate plane.

Challenge Problems

1. Find equations of two planes which intersect in the line

$$\begin{cases} x = 2 \\ y = -1 + t \\ z = 2 + 3t . \end{cases}$$

Explain carefully how you know both the planes contain the line.

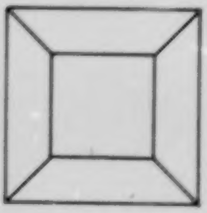
2. Find equations of two planes which intersect in the line

$$\begin{cases} x = 2 \\ y = -1 + t \\ z = -1 . \end{cases}$$

3. Find parametric equations for the "line" L through the "points" $P_0 = (x_0,y_0,z_0,w_0)$ and $P_1 = (x_1,y_1,z_1,w_1)$. Prove that if $P_2 = (x_2,y_2,z_2,w_2)$ is any other "point" on L, then the "line" through P_0 and P_2 contains P_1. Thus there is only one "line" through two given "points".

4. Let $P_0 = (x_0,y_0,z_0,w_0)$. Find the coordinates of the projections of P_0 on the coordinate axes, on the coordinate planes, and on the coordinate hyperplanes. (Before you can do the last part you will have to decide what it means.)

5. A cube in 3-space has an analog in 4-space which is called a tesseract. Make a three-dimensional "picture" of a tesseract. (It may help you to think about the sketch below, in which a cube is drawn in a plane.



The six faces of the cube, which are squares, are represented by two squares and four trapezoids.) In 3-space there is a relationship connecting the numbers of vertices, edges, and faces of a polyhedron. Try to discover this relationship by considering some simple cases. Try to find a corresponding theorem in 4-space.

8-4. The Plane in 3-Space.

In a plane, the set of points equidistant from two distinct points is a line; the equation of a line in 2-space is of first degree. In 3-space, the set of points equidistant from two distinct points is a plane. We review briefly the derivation of the equation of a plane; you may recall it from Intermediate Mathematics.

The point $P = (x,y,z)$ is equidistant from two distinct points $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$, if

$$d(P_1,P) = d(P_2,P) ,$$

or

$$\sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} = \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} .$$

We square both members of the last equation and collect terms, obtaining

$$(1) \quad 2(x_2 - x_1)x + 2(y_2 - y_1)y + 2(z_2 - z_1)z - \left((x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)\right) = 0 .$$

Since $d(P_1, P)$ and $d(P_2, P)$ are positive numbers, this argument can be reversed, and any point $P = (x,y,z)$ whose coordinates satisfy Equation (1) is equidistant from P_1 and P_2.

Equation (1) is a first-degree equation since the coefficients of x, y, and z are not all zero (they could all be zero only if P_1 and P_2 were the same point, but they are distinct).

Thus we have shown that the equation of a plane in three-space is a linear equation of the form

$$(2) \qquad ax + by + cz + d = 0 ,$$

where

$$a = 2(x_2 - x_1) , \quad b = 2(y_2 - y_1) , \quad c = 2(z_2 - z_1) ,$$

and

$$d = -\left((x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)\right) .$$

The proof of the converse--that every equation of the form (2) represents a plane--is left as an exercise.

We note that the coefficients of x, y, and z in Equation (1) are direction numbers of $\overleftrightarrow{P_1P_2}$, a line perpendicular to the plane; hence they are direction numbers of any normal to the plane. We shall extend this idea in Section 8-6. We also note that since $P_1 \neq P_2$, the coefficients a, b, and c are not all zero. The restriction on a, b, c is necessary. Let $a = b = c = 0$. If d is not zero, no triple (x,y,z) satisfies the equation, while if d is zero, every triple satisfies the equation. Neither one of these sets is a plane.

Let us consider certain first-degree equations in which some coefficients are zero. If the equation is of the form $ax = 0$ (or $x = 0$), it represents a plane in which the x-coordinate of every point is zero; clearly this is the yz-plane. In the same way, equations of the other coordinate planes are of the form $by = 0$ (or $y = 0$) and $cz = 0$ (or $z = 0$).

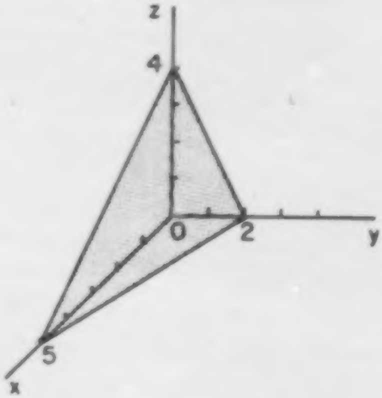
In general, we may find it helpful in visualizing a plane whose equation is given, and in drawing its graph, to find the traces. These are the intersections of the plane with the coordinate planes.

Example 1. Sketch the graph of $4x + 10y + 5z - 20 = 0$.

Solution. To find the trace in the xy-plane we let $z = 0$ in the equation of the plane, obtaining

$$4x + 10y - 20 = 0 .$$

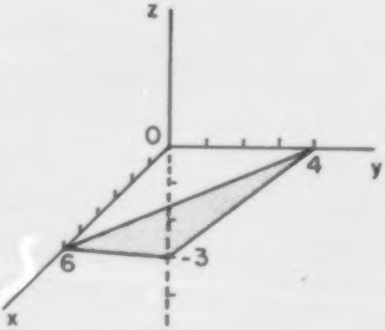
This is the equation of a straight line in the xy-plane.



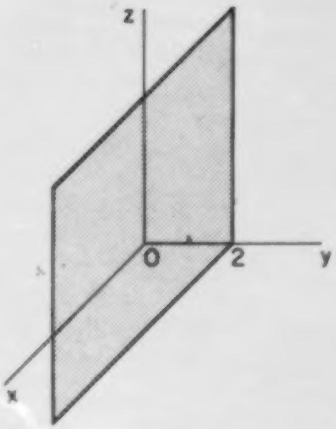
In similar fashion, we find equations of the traces in the yz- and xz-planes ($10y + 5z - 20 = 0$ and $4x + 5y - 20 = 0$ respectively.) The graphs of these lines in the coordinate planes (or the parts of the graphs in one octant) suggest the graph of $4x + 10y + 5z - 20 = 0$.

Example 2. Sketch the graph of $2x + 3y - 4z - 12 = 0$.

Solution. As in Example 2, we find equations of the traces in the xy-, yz-, and xz-planes ($2x + 3y - 12 = 0$, $3y - 4z - 12 = 0$, and $2x - 4z - 12 = 0$ respectively) and then make the sketch.



Example 3. Sketch the graph of $y - 2 = 0$.



Solution. We procede as before, drawing the graphs of $y = 2$, the equation of the traces in the xy- and yz-planes. There is no trace in the xz-plane; to make our representation compatible with our idea of a plane, we complete a parallelogram parallel to the xz-plane.

Since, if two different planes intersect, their intersection is a line, we can represent a line by the equations of any two different planes containing that line. With this in mind, let us look again at what we found in Section 8-3 as the symmetric equations for a line L ,

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m} = \frac{z - z_0}{n} .$$

These equations are equivalent to any pair of the three equations

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m}$$

$$\frac{x - x_0}{\ell} = \frac{z - z_0}{n}$$

$$\frac{y - y_0}{m} = \frac{z - z_0}{n} ,$$

We know from the argument in Section 8-3 that each of the three planes contains L . Furthermore, each one lacks one of the variables. This means that each of the planes is perpendicular to one of the coordinate planes. This follows because, in the first of these three planes, for example, if (x_1,y_1,z_1) is a point in the plane, so also is (x_1,y_1,k) where k has any real value. Thus for any point of the plane, a line perpendicular to the xy-plane through that point is contained in the plane. These symmetric equations represent three planes, each containing the line and each perpendicular to a coordinate plane. These planes are called the projecting planes of L . They are special cases of the projecting cylinders of a curve which will be considered in Chapter 9.

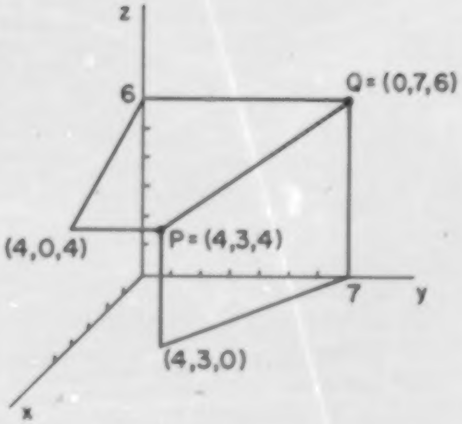


Figure 8-5

Example 4. Sketch the line with equations

$$\frac{x-4}{2} = \frac{y-3}{-2} = \frac{z-4}{-1}$$

by using projecting planes.

Solution. We write the equations of two of the projecting planes,

$$\frac{x-4}{2} = \frac{y-3}{-2}$$

and

$$\frac{x-4}{2} = \frac{z-4}{-1} .$$

These equations may be rewritten as $x + y = 7$ and $x + 2z = 12$. We draw parts of the lines with these equations in the xy- and xz-planes, and complete the sketch as shown in Figure 8-5.

Now we turn to the problem of finding the distance between a point $P_0 = (x_0,y_0,z_0)$ and a plane M with equation

$$ax + by + cz + d = 0 .$$

There is a unique line N, containing P_0, and normal to plane M. If N and M intersect at P_1, the distance between P_0 and M, which we seek, is $d(P_0,P_1)$. We write parametric equations for N, using direction cosines; they are

$$x = x_0 + \lambda t$$
$$y = y_0 + \mu t$$
$$z = z_0 + \nu t .$$

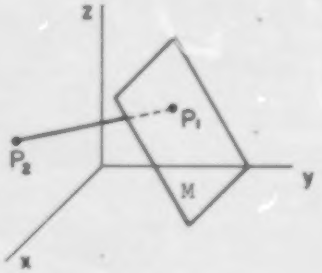


Figure 8-6

Let t_1 represent the particular value t which gives the distance between P_0 and P_1, the point in which N intersects M. Since P_1 is in M, its coordinates satisfy the equation for M;

hence

$$a(x_0 + \lambda t_1) + b(y_0 + \mu t_1) + c(z_0 + \nu t_1) + d = 0 ,$$

or

$$(a\lambda + b\mu + c\nu)t_1 = -(ax_0 + by_0 + cx_0 + d) .$$

If we divide both members of this equation by $\sqrt{a^2 + b^2 + c^2}$ we get

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\lambda + \frac{b}{\sqrt{a^2 + b^2 + c^2}}\mu + \frac{c}{\sqrt{a^2 + b^2 + c^2}}\nu\right)t_1 = -\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} .$$

Since a , b , c are direction numbers for N , $\frac{a}{\sqrt{a^2 + b^2 + c^2}} = \lambda$, $\frac{b}{\sqrt{a^2 + b^2 + c^2}} = \mu$, and $\frac{c}{\sqrt{a^2 + b^2 + c^2}} = \nu$. We substitute λ , μ , ν , and obtain

$$(\lambda^2 + \mu^2 + \nu^2)t_1 = -\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} .$$

But, since λ , μ , and ν are direction cosines, $\lambda^2 + \mu^2 + \nu^2 = 1$; so

$$t_1 = -\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}} ,$$

and (3)

$$d(P_0,P_1) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} .$$

Example 5. Find the distances between $P = (1,-2,3)$ and planes $M_1 = \{(x,y,z) : 3x - 2y + z - 5 = 0\}$ and $M_2 = \{(x,y,z) : x + y = 0\}$.

Solution. Using Equation (3) , we find that

$$d(P_2,P_1) = \frac{|3(1) - 2(-2) + 1(3) - 5|}{\sqrt{9 + 4 + 1}} = \frac{5}{\sqrt{14}} ,$$

and

$$d(P_1,P_2) = \frac{|1(1) + 1(-2)|}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}} .$$

Exercises 8-4

1. Write and simplify the equation of the locus of points equidistant from $A = (-2,3,5)$ and $B = (2,1,-3)$. Check your work by using a different method to find the equation of the plane which is the locus.

2. Follow the instructions in the first exercise, but use $A = (3,1,-4)$ and $B = (2,-3,1)$.

3. Find the intercepts and traces of the planes whose equations are given, and sketch the planes.

   (a) $6x + 4y + 3z - 12 = 0$
   (b) $2x + 5y + z - 10 = 0$
   (c) $4x - 2y - 5z - 10 = 0$
   (d) $3x - 2y + z + 6 = 0$
   (e) $3x - 4y - 12 = 0$
   (f) $5y - 8z + 20 = 0$
   (g) $3x - 6y + 2z = 0$
   (h) $3y - 5z = 0$
   (i) $x - 7 = 0$
   (j) $2z + 9 = 0$

4. Write an equation of the family of planes:

   (a) containing the origin
   (b) parallel to the xy-plane
   (c) parallel to the yz-plane
   (d) parallel to the z-axis
   (e) parallel to the x-axis
   (f) perpendicular to the xz-plane

5. Draw the line determined by the points $A = (5,1,3)$ and $B = (1,4,5)$ by

   (a) using the method described in Exercises 8-2, no. 4; and
   (b) drawing two of the projecting planes.

6. Repeat Exercise 5, using $A = (2,2,3)$ and $B = (0,5,5)$.

7. What is a set of direction numbers for a line perpendicular to the plane $M = \{(x,y,z) : 3x - 2y + 5z - 7 = 0\}$? Write the direction cosines for such a line.

8. Repeat Exercise 7 for the plane $M = \{(x,y,z) : 4x - y + 2 = 0\}$.

9. Find the distance from the point $P = (-1,2,2)$ to each of the planes with equations given in Exercise 3.

10. Repeat Exercise 9 but use the point $P = (1,4,-1)$.

11. Find an equation of the plane through the points

   (a) $(1,2,3)$, $(-1,-1,4)$, $(2,0,1)$
   (b) $(2,1,1)$, $(5,2,3)$, $(-1,-1,-1)$

12. Find an equation of a plane through P and parallel to M if

    (a) $P = (1,2,-3)$; $M = \{(x,y,z) : 3x - 2y + z - 7 = 0\}$

    (b) $P = (-1,2,2)$; $M = \{(x,y,z) : x - 2z + 3 = 0\}$

13. Show that if the x- , y- , and z-intercepts of a plane are a , b , and c respectively, an equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 .$$

14. Write an equation of the plane with x- , y- , and z-intercepts respectively

    (a) $1 , 3 , 4$;

    (b) $-2 , 5 , -3$.

15. Write an equation of a plane containing the point P and the intersection of planes M and N when

    (a) $P = (1,0,2)$, $M = \{(x,y,z) : x - 2y + z - 1 = 0\}$,
    $N = \{(x,y,z) : 2x + y + z + 1 = 0\}$.

    (b) $P = (3,1,-1)$, $M = \{(x,y,z) : x + 3y - 4 = 0\}$,
    $N = \{(x,y,z) : y - 2z + 3 = 0\}$.

16. Show that the four points $A = (1,2,1)$, $B = (2,-1,-4)$, $C = (0,1,2)$, $D = (2,3,0)$ are coplanar.

17. Find an equation of the plane containing the points:

    (a) $(1,-1,1)$, $(2,0,0)$, $(-1,-1,2)$

    (b) $(1,3,5)$, $(2,1,2)$, $(0,-1,-1)$

18. Prove that any equation of the form $ax + by + cz + d = 0$ represents a plane. (This is the converse of the proof at the beginning of this section.)

8-5. Vectors in Space; Components in 3-Space.

For vectors the extension to 3-space is not only natural, but also particularly easy. In your study of Chapter 3 you may have realized that the distinction between parallel and collinear vectors is not as clear as the distinction between parallel and collinear directed segments. Actually, there is no distinction. Because a vector is a set of equivalent directed segments, two vectors which have representatives on parallel lines also have representatives on the same line. In fact, a vector on a line has representatives anywhere on any line parallel to the given line. If $\hat{a}$ is a vector, every point in space is the initial point (or, for that matter, any other point on the line) of a representative of a . This is the basis for the Origin Principle and the Origin-Vector Principle.

For the same reason no two vectors may be noncoplanar. If the representatives of two vectors lie on skew (noncoplanar) lines, they not only have other representatives in a single plane, but also representatives in any other parallel plane. Furthermore, in such a plane they may be represented, of course, by origin-vectors.

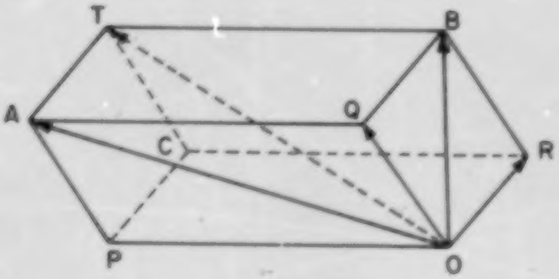
The definitions and properties of operations which involve no more than two vectors, such as addition, scalar multiplication, the distributive laws, and the inner product, apply in space, and may be interpreted geometrically in space. Theorems describing relations between two vectors also apply and may be interpreted in space. If at this point you will reread the definitions, principles, and theorems developed in Section 3-2 through Section 3-5 (pages 91-112), you will see that every statement and proof applies to vectors in space. The figures illustrate the situation in a plane, and in accordance with the Origin-Vector Principle our proofs are in terms of origin-vectors which are coplanar. As our discussion here indicates, our definition of vectors is such that a geometric relationship in space may often be described by vectors in a plane. In general, the vector description of a problem in space frequently may be reduced to a vector illustration in a plane. The illustration in the plane may serve as a simpler guide to the algebraic relations between the vectors. The results obtained may then be applied to the original problem in space. Of course, we must bear in mind that not all sets of vectors are coplanar.

As you reviewed the material in Chapter 3, you may have wondered whether the discussion above justifies the statement that Theorem 3-2, the associative property for vector addition, does apply in space. After all, the theorem states that $\overset{\frown}{P} + (\overset{\frown}{Q} + \overset{\frown}{R}) = (\overset{\frown}{P} + \overset{\frown}{Q}) + \overset{\frown}{R}$, and the three origin-vectors need not be coplanar. Strictly speaking, the assertion is valid, for vector addition is a binary operation: that is, we never add more than two vectors at a time. Therefore, as we perform each step of the proof, we are only adding vectors in a single plane, though the plane we work in may change from step to step in the proof as a whole. Still, the theorem is interesting and illustrative enough to consider as an example.

Example 1. Prove the associative property for vector addition:

$$\overset{\frown}{P} + (\overset{\frown}{Q} + \overset{\frown}{R}) = (\overset{\frown}{P} + \overset{\frown}{Q}) + \overset{\frown}{R} .$$

Proof. In the figure below we illustrate three noncoplanar origin vectors, $\overset{\frown}{P}$, $\overset{\frown}{Q}$, and $\overset{\frown}{R}$. The segment $\overline{AQ}$ is drawn parallel and congruent to $\overline{PO}$ and the segment $\overline{RB}$ is drawn parallel and congruent to $\overline{OQ}$. Each of the quadrilaterals POQA and ORBQ are parallelograms, since in each two opposite sides are parallel and congruent. $\overline{BT}$ is drawn parallel and congruent to $\overline{AQ}$, and thus also to $\overline{PO}$.



$\overline{AT}$ is drawn. Since $\overline{TB}$ and $\overline{AQ}$ are parallel and congruent, quadrilateral AQBT is a parallelogram. Therefore, $\overline{AT}$ is parallel to $\overline{QB}$, and also to $\overline{OR}$. (If $\overline{CR}$ is drawn parallel and congruent to $\overline{PO}$, and $\overline{PC}$ and $\overline{CT}$ are also drawn, the entire figure is a parallelopiped, a prism whose base is a parallelogram region. However, we have not quite proved this here.) Since $\overline{PO}$ and $\overline{TB}$ are parallel and congruent, quadrilateral POBT is a parallelogram. Since $\overline{AT}$ and $\overline{OR}$ are parallel and congruent, quadrilateral ORTA is also a parallelogram.

We have now identified enough parallelograms to enable us to perform the vector additions required in the statement of the associative property.

The left member

$$\overset{\frown}{P} + (\overset{\frown}{Q} + \overset{\frown}{R}) = \overset{\frown}{P} + \overset{\frown}{B} = \overset{\frown}{T} ,$$

since ORBQ and POBT are parallelograms, and the right member

$$(\overset{\frown}{P} + \overset{\frown}{Q}) + \overset{\frown}{R} = \overset{\frown}{A} + \overset{\frown}{R} = \overset{\frown}{T} ,$$

since POQA and ORTA are parallelograms, thus

$$\overset{\frown}{P} + (\overset{\frown}{Q} + \overset{\frown}{R}) = (\overset{\frown}{P} + \overset{\frown}{Q}) + \overset{\frown}{R} .$$

Once a rectangular coordinate system has been introduced in 3-space, we have a one-to-one correspondence between the ordered triples of real numbers and the terminal points of origin-vectors. Thus, if the terminal point of the origin-vector $\widehat{A}$ has coordinates (a_1,a_2,a_3), we may denote $\widehat{A}$ in component form by $[a_1,a_2,a_3]$, where a_1, a_2, and a_3 are the x-, y-, and z- components respectively.

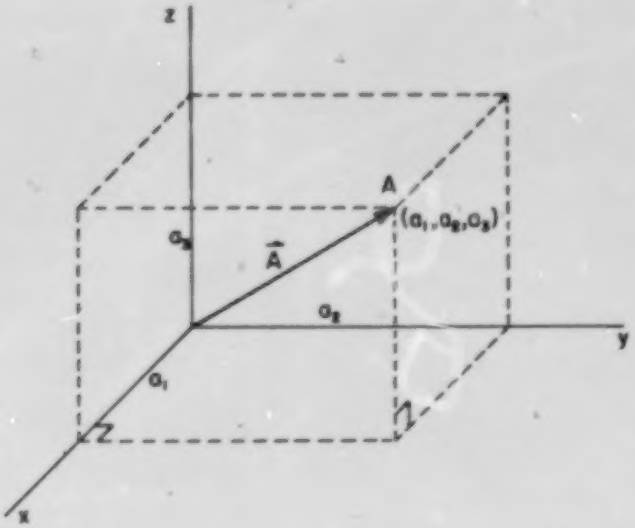


Figure 8-7

It follows from the definition that two vectors $\widehat{a}$ and $\widehat{b}$ are equal if and only if the component forms of their origin-vectors are identical; that is, $\widehat{a} = \widehat{b}$ if and only if $[a_1,a_2,a_3] = [b_1,b_2,b_3]$, and $[a_1,a_2,a_3] = [b_1,b_2,b_3]$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Several theorems in Chapter 3 were proved to hold in the plane using components. We shall restate them here with modifications appropriate to their interpretation in space. We suggest proofs for some and leave the rest as exercises.

THEOREM 8-2. If $\overset{\frown}{A} = [a_1, a_2, a_3]$ and $\overset{\frown}{B} = [b_1, b_2, b_3]$,
$\overset{\frown}{A} + \overset{\frown}{B} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.

We note that if the sum is $\overset{\frown}{X}$, then $\overline{OX}$ and $\overline{AB}$ bisect each other at $\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}\right)$. Thus $\overset{\frown}{X} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$, and $\overset{\frown}{X} = \overset{\frown}{A} + \overset{\frown}{B} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.

THEOREM 8-3. Multiplication of a vector $\overset{\frown}{A}$ by a scalar r is given by $r\overset{\frown}{A} = [ra_1, ra_2, ra_3]$.

The proof is left as an exercise.

THEOREM 8-4. The inner product of two vectors $\overset{\frown}{A}$ and $\overset{\frown}{B}$ is given by $\overset{\frown}{A} \cdot \overset{\frown}{B} = a_1b_1 + a_2b_2 + a_3b_3$.

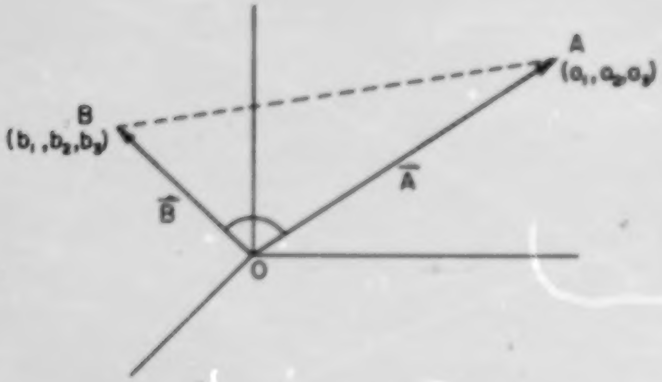


Figure 8-6

By definition $\overset{\frown}{A} \cdot \overset{\frown}{B} = |\overset{\frown}{A}||\overset{\frown}{B}| \cos \theta$; in triangle AOB we see by the Law of Cosines that

$$\cos \theta = \frac{|\overset{\frown}{A}|^2 + |\overset{\frown}{B}|^2 - (d(A,B))^2}{2|\overset{\frown}{A}||\overset{\frown}{B}|}$$

Thus,

$$\overset{\frown}{A} \cdot \overset{\frown}{B} = |\overset{\frown}{A}||\overset{\frown}{B}| \frac{|\overset{\frown}{A}|^2 + |\overset{\frown}{B}|^2 - (d(A,B))^2}{2|\overset{\frown}{A}||\overset{\frown}{B}|}$$

$$= \frac{1}{2}(a_1{}^2 + a_2{}^2 + a_3{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 - ((a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2)$$

$$= \frac{1}{2}(2a_1b_1 + 2a_2b_2 + 2a_3b_3) .$$

$$= a_1b_1 + a_2b_2 + a_3b_3 .$$

THEOREM 8-5. If $\overset{\frown}{X}$, $\overset{\frown}{Y}$, and $\overset{\frown}{Z}$ are any vectors, then

(a) $\overset{\frown}{X} \cdot (\overset{\frown}{Y} + \overset{\frown}{Z}) = \overset{\frown}{X} \cdot \overset{\frown}{Y} + \overset{\frown}{X} \cdot \overset{\frown}{Z}$

(b) $(t\overset{\frown}{X}) \cdot \overset{\frown}{Y} = t(\overset{\frown}{X} \cdot \overset{\frown}{Y})$

Corollary. $\overset{\frown}{X} \cdot (a\overset{\frown}{Y} + b\overset{\frown}{Z}) = a(\overset{\frown}{X} \cdot \overset{\frown}{Y}) + b(\overset{\frown}{X} \cdot \overset{\frown}{Z})$.

The proofs are left as exercises. The other theorems of Chapter 3 were not proved using components and involve no more than two vectors; hence, they apply in 3-space.

Example 2. Find the angle formed by the origin-vectors to the points $A = (2,-3,3)$ and $B = (-1,3,1)$.

Solution. We recognize that the inner product,

$$\overset{\frown}{A} \cdot \overset{\frown}{B} = |\overset{\frown}{A}||\overset{\frown}{B}| \cos \theta ,$$

will help here. Since $\overset{\frown}{A} = [2,-3,3]$ and $\overset{\frown}{B} = [-1,3,1]$, we have

$$2 \cdot (-1) + (-3) \cdot 3 + 3 \cdot 1 = \sqrt{2^2 + (-3)^2 + 3^2}\ \sqrt{(-1)^2 + 3^2 + 1^2} \cos \theta ,$$

$$-8 = \sqrt{22} \cdot \sqrt{11} \cos \theta ,$$

and

$$\cos \theta = \frac{-4\sqrt{2}}{11}$$

$$\approx -.514 .$$

Hence

$$\theta \approx 121^\circ .$$

We recall that any vector expressed in component form in the plane may be resolved into component vectors along the axes. The component vectors in turn may be expressed as scalar multiples of unit vectors. Thus we may resolve a vector $\overrightarrow{A}$ as follows:

$$\begin{aligned}\overrightarrow{A} &= [a_1,a_2,a_3] \\ &= [a_1,0,0] + [0,a_2,0] + [0,0,a_3] \\ &= a_1[1,0,0] + a_2[0,1,0] + a_3[0,0,1] \ .\end{aligned}$$

It is customary to denote the unit vectors $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ by i, j, and k respectively. Since any vector $\overrightarrow{A}$ may be expressed as a linear combination of i, j, and k as

$$\overrightarrow{A} = a_1 i + a_2 j + a_3 k \ .$$

we say that i, j, and k form a basis for 3-space.

The use of vectors gives a concise way of describing a line in 3-space. Let (ℓ,m,n) be a triple of direction numbers of a given line L which passes through the point $P_0(x_0,y_0,z_0)$. Thus a parametric representation of L is

$$\begin{aligned}x &= x_0 + \ell t \\ y &= y_0 + mt \\ z &= z_0 + nt \ .\end{aligned}$$

The vector $\overrightarrow{D} = [\ell,m,n]$ lies on the line L', which has a parametric representation

$$\begin{aligned}x &= \ell t \\ y &= mt \\ z &= nt \ ,\end{aligned}$$

and which is parallel to L. Thus a triple of direction numbers (ℓ,m,n) of a line L determines a vector parallel to L. Furthermore, the point $P(x,y,z)$ lies on L if and only if

$$\overrightarrow{P} = \overrightarrow{P_0} + t\overrightarrow{D} \ .$$

If L is the line which passes through two distinct points $P_0(x_0,y_0,z_0)$ and $P_1(x_1,y_1,z_1)$, then, from Chapter 2, $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ is a triple of direction numbers of L. As we have just seen, this triple of direction numbers determines a vector $\vec{D}$ which is parallel to L. But

$$\vec{D} = [x_1,y_1,z_1] - [x_0,y_0,z_0] = \vec{P}_1 - \vec{P}_0 .$$

Thus, $\vec{P}_1 - \vec{P}_0$ is a vector parallel to the line through P_0 and P_1.

Example 3. Find a vector representation for the line $\overleftrightarrow{P_0P_1}$, where $\vec{P}_0 = 3i + 2j - 4k$ and $\vec{P}_1 = -2i + j + 2k$.

Solution. $P_0 = (3,2,-4)$ and $P_1 = (-2,1,2)$. Hence $\overleftrightarrow{P_0P_1}$ has $(5,1,-6)$ as a triple of direction numbers; $\vec{D} = [5,1,-6]$ is a direction vector for the line. Hence, the vector representation of the line,

$$\vec{P} = \vec{P}_0 + t\vec{D} ,$$

becomes

$$\vec{P} = [3,2,-4] + t[5,1,-6]$$
$$= [3 + 5t , 2 + t , -4 - 6t] ,$$

or

$$\vec{P} = (5t + 3)i + (t + 2)j - (6t + 4)k .$$

Exercises 8-5

1. Let $i = [1,0,0]$, $j = [0,1,0]$, and $k = [0,0,1]$. Find
   - (a) $i \cdot j$
   - (b) $i \cdot k$
   - (c) $j \cdot k$
   - (d) $i \cdot i$
   - (e) $j \cdot j$
   - (f) $k \cdot k$
   - (g) $(4j + 2k) \cdot 5i$
   - (h) $(3i + 2j - k) \cdot (2i + j + k)$

2. Find the cosine of the angle between the two vectors in each part of Exercise 2.

3. Given $\vec{B} = 2i + 2j - k$. Find r such that $|r\vec{B}| = 1$.

4. Let $\vec{A} = [2,3,-1]$, $\vec{B} = [3,-2,1]$, $\vec{C} = [-1,3,-2]$. Find
   - (a) $2\vec{A} + 3\vec{B} - \vec{C}$
   - (b) $\vec{A} - 2\vec{B} + 3\vec{C}$
   - (c) $2(\vec{A} + \vec{B}) - 3(\vec{B} - \vec{C})$
   - (d) $5(\vec{A} - \vec{C}) + 3(\vec{C} - \vec{A})$
   - (e) $3(\vec{A} + \vec{B} - \vec{C}) + 2(\vec{A} - \vec{B} + \vec{C})$
   - (f) $5(\vec{C} - \vec{A} + \vec{B}) - 3(\vec{B} + \vec{A} - \vec{C})$

5. Use values of $\vec{A}$, $\vec{B}$, $\vec{C}$, as in Exercise 4, and find X so that
   - (a) $\vec{A} + \vec{B} = \vec{C} + \vec{X}$
   - (b) $2\vec{A} + 3\vec{B} = 4\vec{C} + 5\vec{X}$
   - (c) $2(\vec{A} - \vec{B}) = 3(\vec{C} - \vec{X})$
   - (d) $\vec{A} + 2\vec{X} = \vec{B} + \vec{C} - \vec{X}$
   - (e) $3(\vec{X} + \vec{B}) = 2(\vec{X} - \vec{C})$
   - (f) $\vec{X} + 2(\vec{X} + \vec{A}) + 3(\vec{X} + \vec{B}) = 0$

6. Use the values of $\vec{A}$, $\vec{B}$, $\vec{C}$, as in Exercise 4, and find
   - (a) $\vec{A} \cdot \vec{B}$
   - (b) $2\vec{A} \cdot 3\vec{B}$
   - (c) $3\vec{A} \cdot (\vec{B} + \vec{C})$
   - (d) $2\vec{B} \cdot (3\vec{A} + 2\vec{C})$
   - (e) $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B})$
   - (f) $(2\vec{B} + 3\vec{C}) \cdot (2\vec{B} - 3\vec{C})$
   - (g) $(3\vec{A} + 5\vec{B}) \cdot (3\vec{B} - 2\vec{C})$
   - (h) $(\vec{A} + \vec{B} - \vec{C}) \cdot (\vec{B} - \vec{A} + \vec{C})$
   - (i) $(2\vec{A} - 3\vec{B} + 4\vec{C}) \cdot (5\vec{A} - 2\vec{C} + 4\vec{B})$
   - (j) $\vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{C} \cdot \vec{C}$

7. Discuss and relate

$$\vec{A} \cdot \vec{A} \,, \; |\vec{A}|^2 \;;\; |\vec{A}|^3 \,, \; \vec{A} \cdot \vec{A} \cdot \vec{A} \,.$$

8. Given $\vec{P} = ai + bj + ck$. Give algebraic and geometric interpretations of $\dfrac{\vec{P}}{|\vec{P}|}$.

9. I[illegible] $= 2i + 3j + 4k$ and $\vec{B} = xi - j + 3k$. Find x such that AOB is a right triangle.

10. Given $\vec{A} = 2i + 3j + 4k$ and $\vec{B} = i + j - k$, find the length of th[illegible] projection of $\vec{A}$ upon $\vec{B}$.

11. Show that the line joining the end points of the vectors $\widehat{A} = 2i + 3j + 4k$ and $\widehat{B} = i - j + 4k$ is parallel to the xy-plane.

12. If $\widehat{c} \perp \widehat{a}$ and $\widehat{c} \perp \widehat{b}$, prove that $\widehat{c} \perp (\widehat{a} + \widehat{b})$.

13. Describe in terms of components all unit vectors perpendicular to the xy-plane.

14. Find a vector $\perp$ to both $\widehat{A} = 2i + 3j + 4k$ and $\widehat{B} = i + j - k$. Note: There are many solutions. Can you find a general solution?

15. Find the measures of the angles of the triangle with vertices at $A = (2,-1,1)$, $B = (1,-3,5)$, $C = (3,-4,-4)$.

16. Find vector representations of the lines passing through $P = (a,b,c) \neq (0,0,0)$ which are perpendicular to $\widehat{P}$.

17. Prove Theorem 8-3.

18. Prove Theorem 8-5 and its Corollary.

8-6. Vector Representations of Planes and Other Sets of Points.

In the first course in geometry plane is an undefined term; its use is described in the postulates. From the postulates we learn that a plane is a set of points and is uniquely determined by three noncollinear points. Further, if two points lie in a plane, then every point of the line containing these points also lies in the plane, and if two different planes intersect, their intersection is a line. A line and a plane were defined to be perpendicular if and only if they intersect and every line lying in the plane and passing through the point of intersection is perpendicular to the given line.

In Section 8-4 we used the fact that in space the locus of points equidistant from two given points is a plane. This led to analytic representations for planes in rectangular coordinates. In this section we shall consider another description of a plane as a locus and develop vector representations for planes.

We let M be a plane and N be a line perpendicular to M at a point P_0. Any other point P, in M, and P_0 determine a line in M, which by definition is perpendicular to N. By a theorem from geometry, every line perpendicular to N at P_0 is contained in M. Thus, we may consider M to be the locus of lines perpendicular to N at P_0. We call N a normal line to the plane.

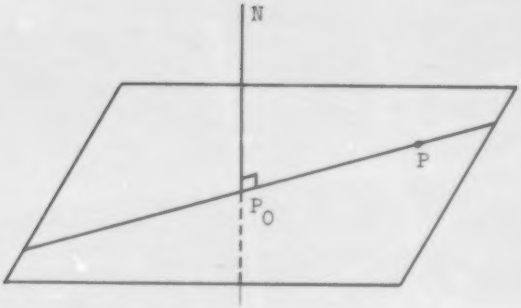


Figure 8-7

The description in terms of perpendicularity suggests a vector representation in terms of the inner product, for if $\overset{\frown}{m}$ is a vector with representatives in M, and $\overset{\frown}{n}$ is a vector with representatives on N, we have $\overset{\frown}{m} \cdot \overset{\frown}{n} = 0$. This will be clearer if we interpret the statement with origin-vectors. The vector $\overset{\frown}{m}$ has a representative $\overset{\frown}{m}_0$ emanating from P_0 which also lies in M. The vector $\overset{\frown}{n}$ also has a representative $\overset{\frown}{n}_0$ emanating from P_0 which lies on N. Hence $\overset{\frown}{m}_0$ and $\overset{\frown}{n}_0$ are perpendicular. Their corresponding origin-vectors $\overset{\frown}{M}$ and $\overset{\frown}{N}$ are perpendicular and $\overset{\frown}{M} \cdot \overset{\frown}{N} = 0$. By the Origin-Vector Principle we may interpret this as $\overset{\frown}{m} \cdot \overset{\frown}{n} = 0$.

To obtain a vector representation of the plane M, we note that if P_1 is a fixed point in M and P is any other point in M, then $\overset{\frown}{P} - \overset{\frown}{P}_1$ is parallel to M. Thus, we may describe the plane M as

$$\{P : (\overset{\frown}{P} - \overset{\frown}{P}_1) \cdot \overset{\frown}{n} = 0\} .$$

We note that P_1 is also in the set.

We recall that it is possible to characterize a line which does not contain the origin in 2-space as the set of points which is perpendicular, or normal, to a directed segment $\overset{\frown}{OP}$ at P. In 3-space we may describe a plane as the set of points which is normal to a directed segment $\overset{\frown}{ON}$, or origin-vector $\overset{\frown}{N}$, at $N \cdot \overset{\frown}{N}$ is called the normal vector of M. If the given point of M is N, then

$$M = \{P : (\overset{\frown}{P} - \overset{\frown}{N}) \cdot \overset{\frown}{N} = 0\} .$$

If we let $P = (x,y,z)$, $|\widehat{N}| = p$, and (λ,μ,ν) be the triple of direction cosines of $\overrightarrow{ON}$, we have

$$\widehat{P} = [x,y,z] ,$$

$$N = (\lambda p , \mu p , \nu p) ,$$

and

$$\widehat{N} = [\lambda p , \mu p , \nu p] = p[\lambda,\mu,\nu] .$$

Thus

$$(\widehat{P} - \widehat{N}) \cdot \widehat{N} = ([x,y,z] - p[\lambda,\mu,\nu]) \cdot p[\lambda,\mu,\nu] = 0 ,$$

which, since $p \neq 0$, is equivalent to

$$[x,y,z] \cdot [\lambda,\mu,\nu] - p[\lambda,\mu,\nu] \cdot [\lambda,\mu,\nu] = 0 ,$$

or

$$\lambda x + \mu y + \nu z - p(\lambda^2 + \mu^2 + \nu^2) = 0$$

Since $\lambda^2 + \mu^2 + \nu^2 = 1$, we have

$$M = \{(x,y,z) : \lambda x + \mu y + \nu z - p = 0\} ,$$

an analytic representation of the plane in terms of the normal form of its equation. We note that (λ,μ,ν) are direction cosines of the normal segment and that p is the distance between the origin and the plane.

Example 1. Find an equation of the plane which is perpendicular to the vector $A = [6,-4,3]$ at the point A .

Solution. We have

$$([x,y,z] - [6,-4,3]) \cdot [6,-4,3] = 0 ,$$

$$[x - 6 , y + 4 , z - 3] \cdot [6,-4,3] = 0 ,$$

and

$$6x - 36 - 4y - 16 + 3z - 9 = 0 ,$$

or

$$6x - 4y + 3z - 61 = 0 .$$

Again we note that the coefficients are direction numbers of normal lines to the plane.

Example 2. Show that if $P_0 = (x_0,y_0,z_0)$ and $P_1 = (x_1,y_1,z_1)$ are two distinct points in a plane with equation $ax + by + cz + d = 0$, then every point of $\overleftrightarrow{P_0P_1}$ is in the plane.

Solution. Any point $P = (x,y,z)$ on line has the parametric representation

$$\begin{aligned} x &= x_0 + (x_1 - x_0)t \\ y &= y_0 + (y_1 - y_0)t \\ z &= z_0 + (z_1 - z_0)t , \end{aligned}$$

and is in the plane if its coordinates satisfy the equation $ax + by + cz + d = 0$. The left member becomes

$$\begin{aligned} & a(x_0 + (x_1 - x_0)t) + b(y_0 + (y_1 - y_0)t) + c(z_0 + (z_1 - z_0)t) + d \\ &= (ax_0 + by_0 + cz_0 + d) + (ax_1 + by_1 + cz_1)t - (ax_0 + by_0 + cz_0)t \\ &= 0 + (-d)t - (-d)t = 0 . \end{aligned}$$

Therefore, any point of the line is contained in the plane.

We may use vectors, as we did in Section 3-6, to describe other sets of points in space.

Example 3. Find a vector representation for the line segment determined by the vectors $\vec{A} = [2,-1,3]$ and $\vec{B} = [-1,4,7]$ in terms of a single parameter p .

Solution. From the development above, $\overline{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p \geq 0$, $q \geq 0$, and $p + q = 1\}$.

Since $p + q = 1$, $q = 1 - p$; since $q \geq 0$, $1 - p \geq 0$ or $p \leq 1$. Since $p > 0$, the combined restriction on p is that $0 \leq p \leq 1$. By substitution,

$$\begin{aligned} p\vec{A} + q\vec{B} &= p[2,-1,3] + (1 - p)[-1,4,7] \quad \text{where } 0 \leq p \leq 1 \\ &= [2p,-p,3p] + [p - 1 , 4 - 4p , 7 - 7p] \quad \text{where } 0 \leq p \leq 1 . \\ &= [3p - 1 , 4 - 5p , 7 - 4p] \quad \text{where } 0 \leq p \leq 1 . \end{aligned}$$

and

$$\overline{AB} = \{X : \vec{X} = [3p - 1 , 4 - 5p , 7 - 4p] \text{ where } 0 \leq p \leq 1\}$$

Example 4. Find a vector representation of the point which divides the directed segment $\overrightarrow{AB}$ in the ratio $\frac{1}{2}$.

Solution.

$$\vec{X} = \frac{2}{1+2}\vec{A} + \frac{1}{1+2}\vec{B}$$

$$= \frac{2}{3}[2,-1,3] + \frac{1}{3}[-1,4,7]$$

$$= [\frac{4}{3}, -\frac{2}{3}, 2] + [-\frac{1}{3}, \frac{4}{3}, \frac{7}{3}]$$

$$= [1, \frac{2}{3}, \frac{13}{3}]$$

Alternatively, if we think of the parameter as a coordinate of the point, then for the desired point $p = \frac{2}{3}$. Substituting this value in the expression obtained in Example 3, we obtain

$$\vec{X} = [3 \cdot \frac{2}{3} - 1 \ , \ 4 - 5 \cdot \frac{2}{3} \ , \ 7 - 4 \cdot \frac{2}{3}]$$

$$= [1, \frac{2}{3}, \frac{13}{3}]$$

Example 5. Find a vector representation for the ray opposite to $\overleftrightarrow{BA}$ in terms of a single parameter q .

Solution. The ray opposite to $\overleftrightarrow{BA} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p \leq 0$ and $p + q = 1\}$. Since

$$p = 1 - q \leq 0 \ ,$$

therefore

$$q \geq 1 \ .$$

$$p\vec{A} + q\vec{B} = (1 - q)[2,-1,3] + q[-1,4,7] \quad \text{where} \quad q \geq 1$$

$$= [2 - 2q \ , \ q - 1 \ , \ 3 - 3q] + [-q, 4q, 7q] \quad \text{where} \quad q \geq 1$$

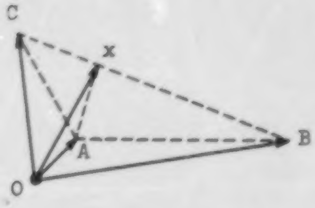
$$= [2 - 3q \ , \ 5q - 1 \ , \ 3 + 4q] \quad \text{where} \quad q \geq 1 \ .$$

The ray opposite to $\overleftrightarrow{BA} = \{X : \vec{X} = [2 - 3q \ , \ 5q - 1 \ , \ 3 + 4q] \ , \text{ where } \ q \geq 1\}$.

Example 4. Suppose $\vec{A}$, $\vec{B}$, and $\vec{C}$ are the vectors whose terminal points are the vertices of a triangle. Can we represent the triangular region, the interior of the triangle, and the triangle itself, in terms of these vectors and two parameters?

Solution. We write $\overline{BC}$ as $\{X : \vec{X} = q\vec{B} + (1 - q)\vec{C}$ where $0 \le q \le 1\}$ as in Example 3 above.

Now the triangular region is the union of the segments $\overline{AX}$ or



$$\{Y : \vec{Y} = p\vec{A} + (1 - p)\vec{X} \text{ where } 0 \le p \le 1\}$$
$$= \{Y : \vec{Y} = p\vec{A} + (1 - p)[q\vec{B} + (1 - q)\vec{C}] \text{ where } 0 \le p \le 1 \text{ and } 0 \le q \le 1\}$$
$$= \{Y : \vec{Y} = p\vec{A} + (1 - p)q\vec{B} + (1 - p)(1 - q)\vec{C} \text{ where } 0 \le p \le 1 \text{ and } 0 \le q \le 1\} .$$

The interior of the triangle ABC will be

$$\{Y : \vec{Y} = p\vec{A} + (1 - p)q\vec{B} + (1 - p)\vec{C} \text{ where } 0 < p < 1 \text{ and } 0 < q < 1\} .$$

The triangle is

$$\{Y : \vec{Y} = p\vec{A} + (1 - p)q\vec{B} + (1 - p)(1 - q)\vec{C} \text{ where } (p = 0 \text{ and } 0 \le q \le 1) \text{ or } (q = 0 \text{ and } 0 \le p \le 1) \text{ or } (q = 1 \text{ and } 0 \le p \le 1)\} .$$

(We can write these results more neatly if we let $r = (1 - p)q$ and $s = (1 - p)(1 - q)$. Then $p + r + s = 1$ and the triangular region is

$\{Y : \vec{Y} = p\vec{A} + r\vec{B} + s\vec{C}$ where p , r , and s are non-negative and $p + r + s = 1\}$

This form is easier to recall.)

Exercises 8-6

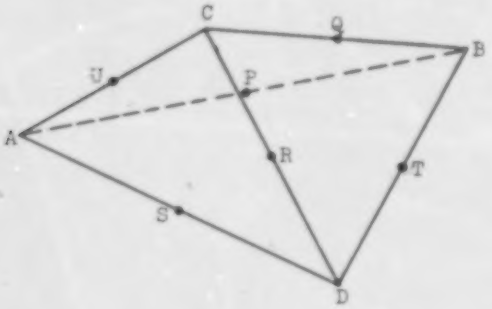
1. Find an equation of the plane which has $[7,-3,5]$ as a normal vector and which contains the point $(0,0,3)$.
2. Find an equation of the plane with the normal vector
   (a) $[2,-3,1]$
   (b) $[-2,4,-7]$
   (c) $[3,-5,4]$
   (d) $[-1,-1,6]$
3. Find the distance from $(0,0,0)$ to the plane
   (a) $2x + 3y - z = 5$
   (b) $5x - 3y + 2z = 8$
   (c) $ax + by + cz = d$

4. In the figure below, consider ABCD to be a 3-dimensional figure. (This is known as a tetrahedron and has 4 faces and 6 edges.)

   (a) Show that the lines through the midpoints of opposite edges are concurrent.

   (b) Show that PTRU and QUST are parallelograms.

   (c) Show that the point of concurrency is the midpoint of each segment.



5. Show that if $P_1 = (x_1, y_1, z_1)$ and $M = \{(x,y,z) : \lambda x + \mu y + \nu z - p = 0\}$, then the distance between P_1 and M is

$$|\lambda x_1 + \mu y_1 + \nu z_1 - p| .$$

6. Find vector representations, in terms of a single parameter, for the sets described below.

   (a) $\overleftrightarrow{AB}$ where $\hat{A} = [4,-7,5]$ and $\hat{B} = [4,2,3]$

   (b) $\overline{AB}$ where $\hat{A} = [3,4,2]$ and $\hat{B} = [-2,3,3]$

   (c) $\overrightarrow{AB}$ where $\hat{A} = [3,4,2]$ and $\hat{B} = [-2,3,3]$

   (d) $\overrightarrow{BA}$ where $\hat{A} = [3,4,2]$ and $\hat{B} = [-2,3,3]$

7. Find the vector representations of the midpoints and trisection points of the following line segments:

   (a) $\overline{AB}$ where $\hat{A} = [0,0,0]$ and $\hat{B} = [6,12,15]$

   (b) $\overline{AB}$ where $\hat{A} = [-3,2,7]$ and $\hat{B} = [10,-11,19]$

   (c) $\overline{AB}$ where $\hat{A} = [a_1,a_2,a_3]$ and $\hat{B} = [b_1,b_2.b_3]$

8. Find the vector representations of the points which divide the directed segment $\overrightarrow{PQ}$ in the ratio $\frac{r}{s}$ where:

   (a) $\overrightarrow{P} = [-3,-2,-1]$, $\overrightarrow{Q} = [3,2,1]$, and $\frac{r}{s} = 1$

   (b) $\overrightarrow{P} = [-1,4,-8]$, $\overrightarrow{Q} = [9,-5,7]$, and $\frac{r}{s} = \frac{1}{5}$

   (c) $\overrightarrow{R} = [2,3,1]$, $\overrightarrow{S} = [1,-2,4]$, and $\frac{r}{s} = \frac{3}{1}$

9. Given the triangle ABC with $\overrightarrow{A} = [2,3,1]$, $\overrightarrow{B} = [-1,2,4]$, and $\overrightarrow{C} = [1,4,-2]$.

   (a) Describe the triangular region, its interior, and the triangle itself, using these vectors and two parameters.

   (b) Show that $[1,3,1]$ is a vector whose terminal point is an interior point of the triangle.

   (c) Show that $[-4,-5,-6]$ is a vector whose terminal point is an exterior point of the triangle.

Challenge Problem

1. Given the four vectors $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, and $\overrightarrow{D}$, whose terminal points are not coplanar, find an expression for the tetrahedral region ABCD in terms of these vectors and three parameters.

8-7. Summary.

We have extended the rectangular coordinate system to 3-space and have considered the analytic and vector representations of lines and planes in 3-space. In Chapter 9 we shall consider the representation and sketching of other curves and surfaces. We shall also consider two extensions of polar coordinates to 3-space.

We have also suggested that we may interpret algebraic relationships in four variables in a 4-space, which may be helpful even though we cannot visualize it. The extension is, of course, possible to spaces of more dimensions. We are in a position to make several conjectures based on our observations in 2-space and 3-space. In 2-space the general linear equation in 2 variables describes a line, a one-dimensional figure; in 3-space the general linear equation in 3 variables describes a plane, a 2-dimensional figure. Thus, in n-space we might expect the general linear equation in n-variables to describe a figure with n-1 dimensions.

In 2-space we are able to describe a line either by a linear equation or by a parametric representation in one parameter; in 3-space we still have the parametric representation of a line in one parameter, but the alternative is the common solution of two linear equations, which is awkward. Some of the later exercises show that we may also describe regions in a plane by a parametric representation in two parameters. Our conjecture might be that in spaces with enough dimensions we may describe one-dimensional figures with parametric representations in one parameter, 2 dimensional figures with parametric representations in two parameters, and, in general, n-dimensional figures with parametric representations in n parameters.

Review Exercises

In Exercises 1 to 8, write an equation of the locus of a point which satisfies the stated conditions.

1. A point 5 units above the xy-plane.
2. A point 5 units from the yz-plane.
3. A point equidistant from the xy- and the yz-planes.
4. A point 2 units from the x-axis.
5. A point a units from the origin.
6. A point r units from the point $(2,-1,0)$.
7. A point equidistant from the point $(1,2,3)$ and the plane with equation $z = 2$.
8. A point that lies in the plane determined by the points $(3,1,2)$, $(1,2,3)$, $(2,2,2)$.

Sketch the graph of the equations in Exercises 9 to 14.

9. $x + y - 4 = 0$
10. $2z - 7 = 0$
11. $4x + 9y - 6z + 36 = 0$
12. $x - y + z + 3 = 0$
13. $x = 5 - 3t, y = 2 + t, z = 3 - 4t$
14. $\frac{x - 5}{-3} = \frac{y - 2}{-2} = \frac{z - 3}{4}$

In Exercises 15-20, graph and describe the geometric representation in one-space and 2-space, and discuss a possible meaning in 3-space.

15. $\{x : x - 3 = 0\}$

16. $\{x : -1 < x < 3\}$

17. $\{x : |x| - 3 = 0\}$

18. $\{x : |x| \geq 3\}$

19. $\{x : |x| \leq 5\}$

20. $\{x : x(x - 1)(x + 2) = 0\}$

21. Graph and describe R_1 , R_2 , and R_3 for one space, 2-space, and 3-space if

$R_1 = \{(x,y) : |x| < 2\}$, $R_2 = \{(x,y) : |y| < 2\}$, $R_3 = R_1 \cap R_2$.

22. Discuss Exercise 21 if $<$ is changed to $\leq$. What geometric interpretation can you give for $R_1 \cup R_2$?

23. Graph and describe $\{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$. What is the graph if $\leq$ is changed to $<$?

In Exercises 24 to 26, use the four points: $A(-2,1,3)$, $B(3,1,-2)$, $C(2,3,-1)$, $D(1,-3,2)$, and the four planes:

$M_1 : 2x - 3y + z + 4 = 0$, $M_2 : 3x - y + 2z - 3 = 0$, $M_3 : x + 2y - 3z + 2 = 0$

$M_4 : -x + y + z - 1 = 0$.

24. Find the distance from each of the points A , B , C , D , O to each of the planes:

(a) M_1

(b) M_2

(c) M_3

(d) M_4

25. Find, in symmetric form, equations of the lines determined by:

(a) $\{M_1,M_2\}$

(b) $\{M_1,M_3\}$

(c) $\{M_1,M_4\}$

(d) $\{M_2,M_3\}$

(e) $\{M_2,M_4\}$

(f) $\{M_3,M_4\}$

26. Find parametric equations for each of the lines referred to in Problem 25.

27. Show that the space quadrilateral ABCD , where $A = (-2,3,2)$, $B = (-4,5,8)$, $C = (1,1,4)$, $D = (3,-1,-2)$, is a parallelogram.

28. Show that the medians of triangle ABC , where $A = (0,0,0)$, $B = (2,4,6)$, $C = (-4,2,-8)$, are concurrent.

29. For what value of a are the points $(3,2,3)$, $(1,-4,2)$, $(2,14,5)$ collinear?

30. If $(2,1,4)$, $(0,4,-2)$, $(a,-2,-4)$ are the vertices of a triangle with a right angle at vertex $(0,4,-2)$, find a .

Chapter 9

QUADRIC SURFACES

9-1. What Is a Quadric Surface?

If you know what is meant by "quadratic equation," you might guess what is meant by "quadric surface". The locus, if one exists, of an equation of the second degree in rectangular coordinates for 3-space is called a quadric surface. Each of these surfaces has an important property: all plane sections are conics. There are many surfaces other than quadric surfaces, and there are more quadric surfaces than the ones we shall introduce. We shall limit our discussion to the most useful and easily recognized ones. You will recognize spheres, cones, and cylinders. Some of the other surfaces may be less familiar to you, but, inasmuch as all intersections of these surfaces with planes are conic sections, you should have little difficulty visualizing even those quadric surfaces which are new to you.

When we apply mathematics to physical problems, we find that a drawing which depicts the physical relations in the problem can be useful. Our principal aim in this chapter is to develop methods for visualizing surfaces and curves in 3-space. Such configurations frequently occur in science and calculus courses. We shall give directions involving only simple figures and equations, but the methods are general and can be extended to more complicated cases. We also shall indicate how equations representing quadric surfaces or space curves may be simplified.

Some ability in the sketching of geometric figures is required in this chapter; you must make drawings of three-dimensional objects on a two-dimensional surface. Also, we shall rely heavily upon the material which you learned in Chapters 5, 6, and 7.

9-2. Spheres and Ellipsoids.

You are familiar with the graph of the points in a plane at a given distance from a given point, and you also know an equation of this graph. If the given point is taken as the origin and the given distance is 4 , the equation is

$$x^2 + y^2 = 16 .$$

Now suppose we consider this same problem in 3-space. You know that the locus is a sphere of radius 4 , but let us proceed as we would if you did not know this. We shall use various methods to "discover" the shape of this familiar surface. Later you will use the same methods to find the shape of unfamiliar surfaces.

A sphere is defined as the set of points each of which is at a given distance from a given point. It always will be possible to select this given point (the center) as the origin of a rectangular coordinate system. Such a choice will simplify the algebraic representation of the sphere.

We wish to examine the set of points, each of which is a distance 4 from the origin, $0 = (0,0,0)$. For each such point $P = (x,y,z)$, the condition is

$$\sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = 4$$

or (1) $$x^2 + y^2 + z^2 = 16 .$$

An attempt to visualize this sphere by plotting points, such as $(2,3,\sqrt{3})$, $(1,\sqrt{6},3)$, $(\sqrt{2},-3,\sqrt{5})$, not only is tedious but, even when a great many points have been plotted, does not reveal the sphere we expect.

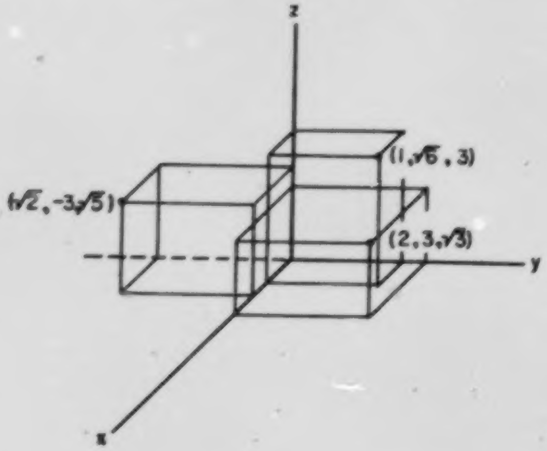


Figure 9-1

It is more illuminating to exploit the similarity between the equation of a sphere and the equation of a circle. For instance, the equation

$$y^2 + z^2 = 16 \tag{2}$$

not only closely resembles our equation (1) of the sphere under discussion, but Equation (2) represents a part of this sphere. It represents, of course, the intersection of the sphere and the yz-plane $(x = 0)$ shown in Figure 9-2. The intersection of a quadric surface and a coordinate plane is called a trace.

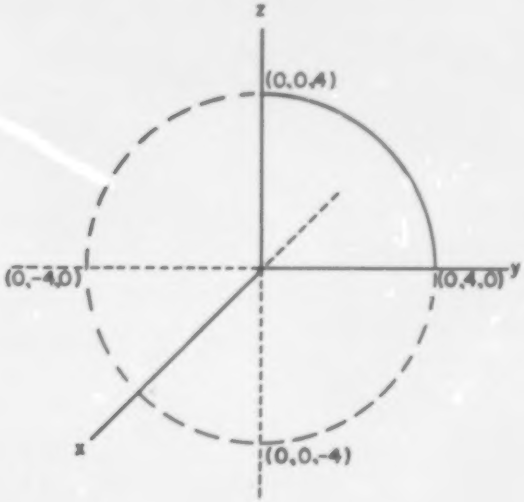


Figure 9-2

The algebraic representation of this trace if the simultaneous solution of Equation (1) and $x = 0$. The traces in the other coordinate planes are found by taking $y = 0$ and $z = 0$. We show in the figure only those parts of traces which are in the boundaries of the first octant.

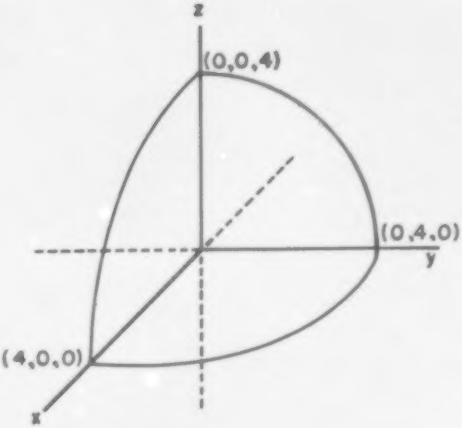


Figure 9-3

In some problems we need help in drawing the traces. In this event we locate the intercepts· the points of intersection of the surface with the coordinate axes. For Equation (1) the values are 4 and -4 on each axis.

Once the traces are indicated, as in Figure 9-3, we begin to see the shape of the surface. Next we investigate the shape of the rest of the surface by slicing it and looking at each slice. Such slices are called sections; they are the curves formed by the surface and planes cutting it. The traces, of course, are special cases of sections. Let us make our slices parallel to the xy-plane. An equation of the parallel plane one unit above the xy-plane is $z = 1$; we substitute for z in Equation (1), which becomes

$$x^2 + y^2 + 1 = 16 ,$$

or

$$x^2 + y^2 = 15 .$$

We see that this is an equation of a circle in a plane parallel to the xy-plane, with radius $\sqrt{15} \approx 3.9$, and with its center on the z-axis; we add to the figure, in the plane $z = 1$, the part of the circle in the first octant. We continue in this fashion, letting z assume the values 2 and 3 . Each section is a circle, and the radii are approximately 3.5 and 2.6 , respectively. We have added parts of these circles in Figure 9-4. When $z = 4$ we have

$$x^2 + y^2 = 0 ,$$

which represents the point $(0,0,4)$. For any value of z larger than 4 , there is no locus.

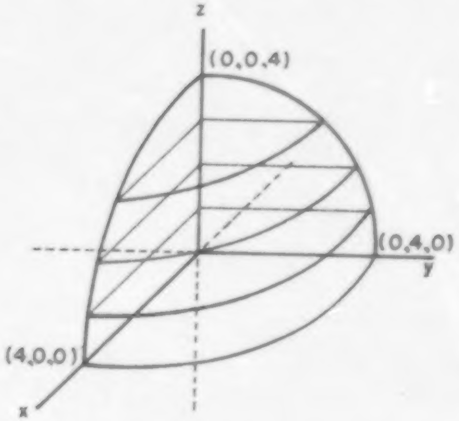


Figure 9-4

Now we consider sections parallel to the yz-plane, giving the same numerical values to x that we gave to z . Again we find that the sections are circles, which we may add to our drawing (Figure 9-5). We might also investigate sections parallel to the xz-plane if this appears to aid our visualization.

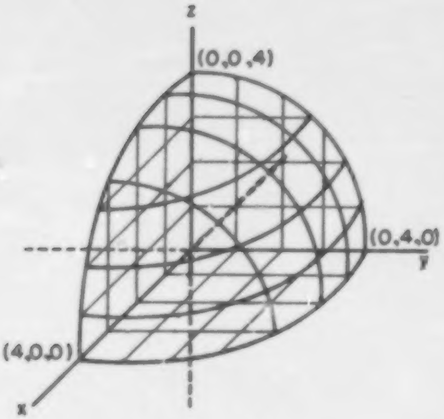


Figure 9-5

This has probably seemed a slow and labored procedure to get a drawing of such a familiar surface as the sphere, but we hope that you will now be able to apply the same methods to other equations in order to visualize and draw the surfaces they represent.

When graphing in three dimensions it is helpful, as it was in two, to investigate symmetry. The definitions of point-symmetry and line-symmetry given in Section 6-2 hold for 3-space, but a more useful idea is that of symmetry with respect to a plane. A set of points S is symmetric with respect to a fixed plane M if and only if for each point P of S there is a corresponding point P' of S such that M is the perpendicular bisector of $\overline{PP'}$. Here we shall investigate symmetry only with respect to the coordinate planes. We list the tests: a graph will be symmetric with respect to the $\begin{Bmatrix}\text{xy-plane}\\ \text{yz-plane}\\ \text{xz-plane}\end{Bmatrix}$ if, whenever (x_1, y_1, z_1) is on the graph, so also is $\begin{Bmatrix}(x_1, y_1 - z_1)\\ (-x_1, y_1, z_1)\\ (x_1, -y_1, z_1)\end{Bmatrix}$.

If a surface is symmetric with respect to all three coordinate planes, it is also symmetric with respect to the origin and each axis. A sphere, of course, meets all these tests for symmetry.

When a surface is symmetric with respect to all three coordinate planes, the part of it in any octant is repeated in all the other octants. In such cases we need draw only that part in the first octant, since this makes our drawing less complicated.

The sphere we have been considering has its center at the origin; the equation for such a sphere can always be written in the form

$$x^2 + y^2 + z^2 = a^2 \tag{3}$$

where $|a|$ is the radius. Note that the terms containing x, y, z all have the coefficient 1.

Consider the equation

$$4x^2 + y^2 + 4z^2 = 100 . \tag{4}$$

What quadric surface does this represent? We begin, as before, by drawing the traces. To find the trace in the yz-plane, we let $x = 0$ in Equation (4), obtaining $\frac{y^2}{100} + \frac{z^2}{25} = 1$. We recognize that this trace is an ellipse, as shown in Figure 9-6. When we let $z = 0$, we again obtain an ellipse. However, when $y = 0$, $x^2 + z^2 = 25$; the trace is a circle. Again we shall picture only those portions of the traces lying in the boundaries of the first octant. These are shown in Figure 9-7.

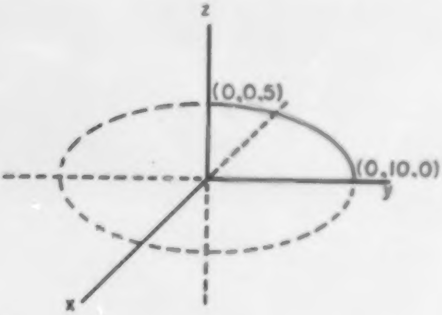


Figure 9-6

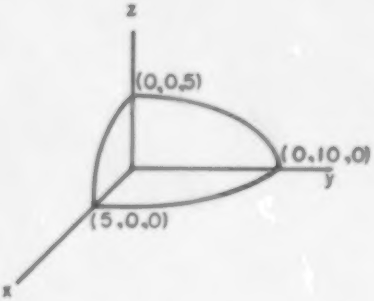


Figure 9-7

Now we find the sections as before; those parallel to the xy- and yz-planes are ellipses; the ones parallel to the xz-plane are circles. It is common practice to select just one set of sections to illuminate the drawing; if one set consists of circles, this is the usual choice. These sections are shown in Figure 9-8.

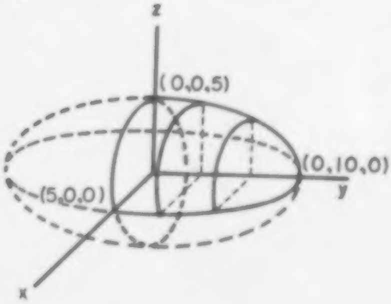


Figure 9-8

The surface we have been sketching belongs to a class called ellipsoids. They are so named because the sections parallel to the coordinates planes are ellipses (or circles, which may be considered special cases of ellipses). These surfaces have equations of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 , \qquad (5)$$

where the numbers $\pm a$, $\pm b$, $\pm c$, are the x- , y- , z- intercepts respectively. The segments of the axes joining the intercept points are called axes of the ellipsoid.

If two of the axes of an ellipsoid have equal length, the surface is called a spheroid, because it resembles a sphere. These are of two kinds. If the third axis is longer than the others as is illustrated in Figure 9-8, the spheroid is called a prolate spheroid and resembles a football or a watermelon. If the third axis is shorter than the other two, the surface is called an oblate spheroid and appears flattened like the earth or a "Yo-Yo" top.

When $a = b = c$ in Equation (5), we have the equation of a sphere. A sphere, then, is a special kind of ellipsoid in much the same sense that a circle is a special kind of ellipse. Before we conclude this section we should ask again, "What quadric surface does Equation (4) represent"? Following what is a good general procedure, you should write Equation (4) in the form of Equation (5) and then name the surface according to the above descriptions.

Exercises 9-2

In Exercises 1 to 12, discuss and sketch the surface represented. Include intercepts, traces, and the name of the surface. Draw several of the sections parallel to one of the coordinate planes.

1. $x^2 + y^2 + z^2 = 25$
2. $4x^2 + 4y^2 + 4z^2 = 9$
3. $9x^2 + 9y^2 + 9z^2 = 0$
4. $9x^2 + 4y^2 + 9z^2 = 36$
5. $9x^2 + 9y^2 + 4z^2 = 36$
6. $4x^2 + 25y^2 + 25z^2 = 100$
7. $4x^2 + 9y^2 + 4z^2 = 36$
8. $9x^2 + 9y^2 + 25z^2 = 225$
9. $9x^2 + 25y^2 + 25z^2 = 225$
10. $4x^2 + 9y^2 + 16z^2 = 144$
11. $9x^2 + 4y^2 + 16z^2 = 144$
12. $16x^2 + 9y^2 + 4z^2 = 144$

13. Use the definition of sphere to write an equation of a sphere with center (x_0, y_0, z_0) and radius r.

14. Show that the equation you obtained in Exercise 13 can always be written in the form

$$x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0 .$$

Does every equation written in this form represent a sphere? Justify your answer.

15. Find, in the form in Exercise 14, equations of the spheres with the given center (C) and radius (r).

(a) $C = (2,1,3)$, $r = 5$ (d) $C = (\frac{1}{3}, -1, \frac{1}{2})$, $r = 1$

(b) $C = (0,-1,2)$, $r = 2$ (e) $C = (\frac{1}{2}, \frac{1}{4}, -\frac{1}{2})$, $r = \frac{1}{2}$

(c) $C = (1,3,-2)$, $r = \sqrt{2}$ (f) $C = (1.5, -.5, 2.5)$, $r = 3$

16. Determine whether the following equations represent spheres. For each sphere, give the radius and the coordinates of the center.

(a) $3x^2 + 3y^2 + 3z^2 - 9 = 0$

(b) $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

(c) $x^2 + y^2 + z^2 - 4y + 2z - 20 = 0$

(d) $x^2 + y^2 + z^2 + 6x - 8y + 14z + 72 = 0$

(e) $x^2 + y^2 + z^2 + 4x - 6y + 13 = 0$

(f) $x^2 + y^2 + z^2 - 2x + 6y + 14 = 0$

(g) $36x^2 + 36y^2 + 36z^2 - 36x - 48y + 12z + 52 = 0$

(h) $16x^2 + 16y^2 + 16z^2 - 24x - 64y - 16z + 41 = 0$

17. If $A = (1,2,3)$ and $B = (-1,0,7)$, what is an equation of the sphere that has $\overline{AB}$ as diameter?

18. Write an equation of an ellipsoid with x- , y- , and z-intercepts ± 3 , ± 7 , ± 5 , respectively.

Challenge Problems

1. Write an equation of an ellipsoid with center at the point $(3,-1,2)$, and with axes parallel to the x- , y- , and z-axes and of lengths 12 , 8 , and 24 respectively.

2. Points $P = (0,3,1)$, $Q = (-2,0,2)$, $R = (1,1,4)$, and $S = (-3,3,2)$ are points of a sphere. What is an equation of the sphere? Will any four distinct points determine a sphere?

9-3. The Paraboloid and the Hyperboloid.

What is the locus of a point equidistant from a given point F and a given plane M ? We shall assume that the distance from F to M is 4 . The geometric condition for the locus is similar to the one which defines a parabola. With this in mind we let the line through F perpendicular to M be the y-axis and let the origin be the midpoint of the normal segment from F to M . Then $F = (0,2,0)$ and the equation of M is $y + 2 = 0$. The required point $P = (x,y,z)$ must meet the condition

$$\sqrt{x^2 + (y - 2)^2 + z^2} = \left|\frac{y + 2}{\sqrt{1}}\right| .$$

Squaring, we have $x^2 + y^2 - 4y + 4 + z^2 = y^2 + 4y + 4$;

hence (1) $$x^2 + z^2 = 8y$$

is an equation for the locus.

Now we must decide what the graph of this equation looks like. We shall use the same methods we applied to the equation of the sphere. If we look for intercepts, we find that the only intersection of the surface with the axes is the origin, $(0,0,0)$. The trace in the xy-plane is the parabola $x^2 = 8y$; in the yz-plane, the parabola $z^2 = 8y$. The trace in the xz-plane is the single point O , given by the equation $x^2 + z^2 = 0$. We notice that in Equation (1) y cannot have negative values; hence no part of the surface is to the left of the xz-plane.

We next investigate the sections parallel to the xz-plane. When $y = 1$, we have $x^2 + z^2 = 8$, a circle with radius $2\sqrt{2}$. For $y = 2$, we have a circle of radius 4 , and so on. Thus the surface may be thought of as formed by a succession of circles, beginning with the point-circle and with radius increasing without limit as y increases. This bullet-shaped surface (Figure 9-9) is called a paraboloid. It is also called a paraboloid of revolution, as it may be generated by revolving a parabola about its axis. The reflector usually called a parabolic reflector is really a paraboloid.

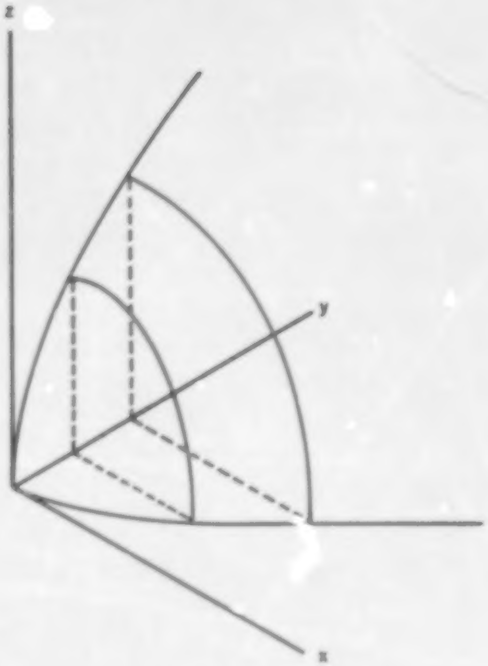


Figure 9-9

A more general equation of a parabolid is of the form

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = by \ . \tag{2}$$

The traces of this surface in the xy- and yz-planes are parabolas, but the sections parallel to the xz-plane are ellipses or circles. This surface is called an elliptic paraboloid.

We turn now to the equation

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{25} = 1 \ , \tag{3}$$

and find that the x- and y-intercepts are ± 2 and ± 3 respectively, but that there are no z-intercepts. The trace in the xy-plane is an ellipse; in the other coordinate planes the traces are hyperbolas. Since ellipses are easier to draw than hyperbolas, let us make our sections parallel to the xy-plane. When $z = 1$ we have

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 + \frac{1}{25} ,$$

representing an ellipse very much like the one which is a trace in the xy-plane. We continue, finding that for numerically larger values of z the sections will be ellipses with increasingly larger intercepts. This surface (Figure 9-10) is called a hyperboloid of one sheet, or an elliptic hyperboloid. Its equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 . \tag{4}$$

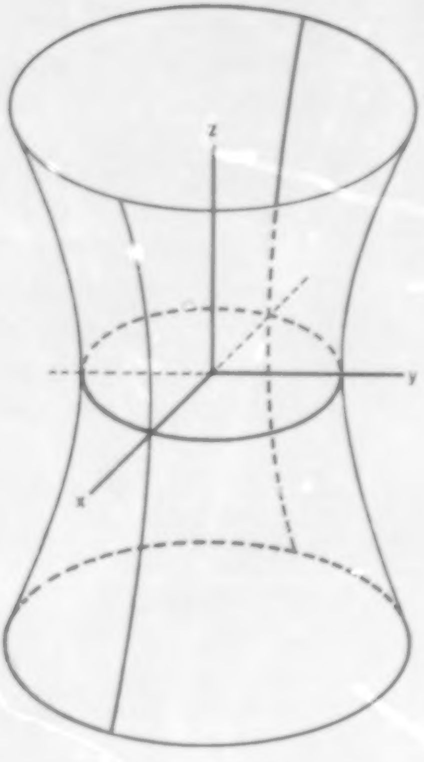


Figure 9-10

Next we consider the equation

$$-\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = 1 . \tag{5}$$

Here there are no x- or y-intercepts; the z-intercepts are ± 5 . The traces in the yz- and xz-planes are hyperbolas. Again we make our sections parallel to the xy-plane. If we write the Equation (5) in the form

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{z^2}{25} - 1 ,$$

we see that when $|z| < 5$ there are no real values of x or y .

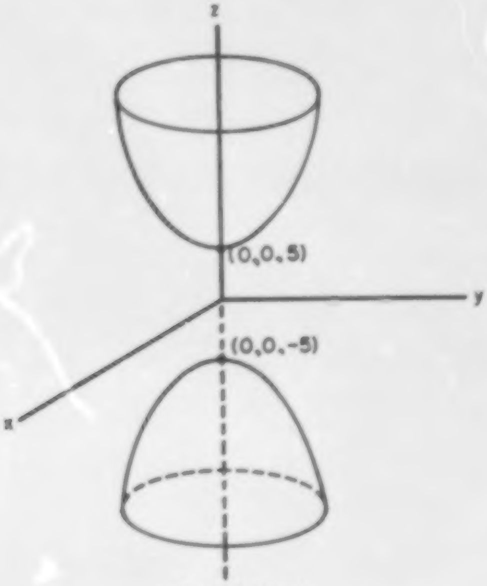


Figure 9-11

When $z = 5$ the section is the point $(0,0,5)$; for $z = -5$, we have the point $(0,0,-5)$. For $|z| > 5$ the sections are ellipses, whose axes increase as $|z|$ increases. Thus our surface may be thought of as two separate piles of ellipses. It is called a hyperboloid (or elliptic hyperboloid) of two sheets.

Exercises 9-3

Discuss and sketch the surfaces represented by the equations in Exercises 1 to 12.

1. $y^2 + z^2 = 4x$
2. $x^2 + y^2 = 16z$
3. $4x^2 + 4z^2 = 16y$
4. $4x^2 + 9z^2 = 144y$
5. $9x^2 + 4z^2 = 144y$
6. $9y^2 + 4z^2 = 144x$
7. $9x^2 + 9y^2 - z^2 = 36$
8. $9x^2 - 4y^2 + 9z^2 = 36$
9. $x^2 - 9y^2 + 4z^2 = 36$
10. $4x^2 - 25y^2 + 4z^2 = 100$
11. $4x^2 - 9y^2 + z^2 = 144$
12. $x^2 - y^2 + z^2 - 1 = 0$

13. We observed that, for the hyperboloid whose graph is given by Equation (3), the sections parallel to the xy-plane are ellipses. Prove that these ellipses have the same eccentricity.

Challenge Problems

The surfaces represented by the following equations are called hyperbolic paraboloids. Discuss and sketch them.

1. $4x^2 - 9y^2 = 36z$.
2. $16y^2 - 9x^2 = 144z$.
3. $y^2 - z^2 = x$.

9-4. Cylinders.

Equations of the quadric surfaces which we have investigated have contained all three variables. What if an equation contains only two variables? Suppose the equation is

$$x^2 + y^2 = 25 . \tag{1}$$

We find the x- and y-intercepts, and note that there are no z-intercepts. The trace in the xy-plane is a circle of radius 5 with the center at O ; in each of the other coordinate planes it is two straight lines, parallel to the coordinate axis. The sections parallel to the xy-plane are all circles of radius 5 with their centers on the z-axis. From Figure 9-12 we recognize the surface as a cylinder.

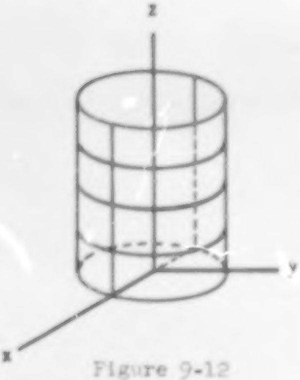


Figure 9-12

A cylindrical surface, or cylinder, is the surface formed when a line moves in space so that it always has the same direction numbers and intersects a fixed plane curve. The plane curve is called a directrix; the lines are called generators or elements. A part of such a surface is shown in

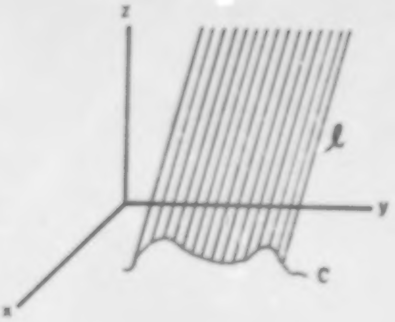


Figure 9-13

Figure 9-13; the curve c in the xy-plane is a directrix, the line ℓ an element. For the circular cyliner in Figure 9-12, any one of the circles we have drawn might be considered a directrix, and any of the lines of the cylinder an element.

We shall restrict our examples to cylinders with elements parallel to an axis. In such cases one of the variables is missing from the equation. For example, we shall consider the equation

$$(2) \qquad \frac{x^2}{36} + \frac{z^2}{9} = 1 .$$

Let us see if we can show that this surface satisfies our definition of a cylinder. If it is a cylinder then the trace in the xz-plane, the ellipse with equations

$$\frac{x^2}{36} + \frac{z^2}{9} = 1 , \; y = 0 ,$$

must be a directrix. We select any point of this ellipse, say $P = (4,0,\sqrt{5})$. We find that for any value y , the point $(4,y,\sqrt{5})$ is a point of the surface. All such points lie on the line ℓ perpendicular to the xz-plane at P ; hence ℓ is an element of the cylinder.

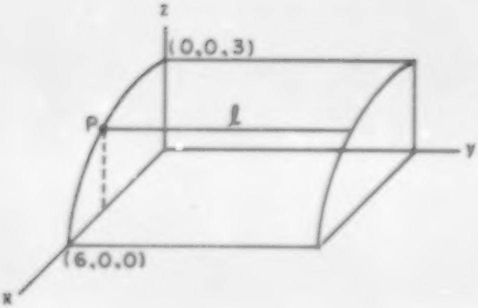


Figure 9-14

Not all cylinders are quadric surfaces. A plane may be considered a cylinder, since one of any two intersecting lines in it may serve as directrix and the other as an element. Other examples of cylinders are the graphs of such equations as $z = \sin y$ and $y = e^x$. You might sketch one of these cylinders.

Exercises 9-4

Discuss and sketch the cylinders represented by equations 1 to 10 .

1. $x^2 + y^2 = 64$
2. $x^2 + z^2 = 25$
3. $y^2 + z^2 = 36$
4. $4x^2 + 9y^2 = 36$
5. $9x^2 + 4z^2 = 36$
6. $4y^2 + 9z^2 = 36$
7. $25x^2 + 144y^2 = 3600$
8. $144x^2 + 25z^2 = 3600$
9. $9x^2 - 4y^2 = 1$
10. $9x^2 - 25y^2 = 1$

11. Write an equation for the locus of points
    - (a) at distance 9 from the x-axis
    - (b) at distance 6 from the y-axis
    - (c) at distance 4 from the z-axis

12. Write an equation for each of the cylinders discribed below.
    - (a) Axis is the x-axis, trace in the yz-plane is a circle of radius 3 .
    - (b) Axis is the y-axis, trace in the xz-plane is a circle of radius 5 .
    - (c) Axis is the z-axis, trace in the xy-plane is a circle of radius 10 .

13. A line moves so that it is always parallel to the y-axis and 10 units from it. What is an equation of its locus?

14. A line moves so that it is always parallel to the x-axis and 12 units from it. What is an equation of its locus?

15. The circle with equations

$$x^2 + z^2 = 4 \text{ , } y = 0$$

is the directrix of a cylinder, and a line parallel to the y-axis is an element. What is an equation of the cylinder?

16. Write an equation of the cylinder with the ellipse with equations

$$25y^2 + 4z^2 = 100 \text{ , } x = 0$$

as directrix, and a line perpendicular to the yz-plane at a vertex of the ellipse as an element.

Challenge Problems

Discuss and sketch the cylinders represented by Equations 1 to 8 .

1. $x^2 = 4z$
2. $y^2 = z$
3. $y^2 - z^2 + 1 = 0$
4. $xy = 12$
5. $x^2 + z^2 - 6z = 7$
6. $x^2 + y^2 + 2x - 4y = 4$
7. $z = \sin x$
8. $y = \cos z$
9. Write an equation for the cylinder with axis parallel to the x-axis, and with trace in the yz-plane a circle of radius 4 and center at $(0,-2,5)$. Sketch the cylinder.

9-5. The Cone.

Let us investigate the surface whose equation is

$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{9} = 0 . \tag{1}$$

When we look for intercepts and the trace in the xy-plane, we find only the point $O = (0,0,0)$. If $x = 0$, Equation (1) becomes

$$\frac{y^2}{4} - \frac{z^2}{9} = 0 ;$$

the trace in the yz-plane is the union of two intersecting lines. So is the trace in the xz-plane.

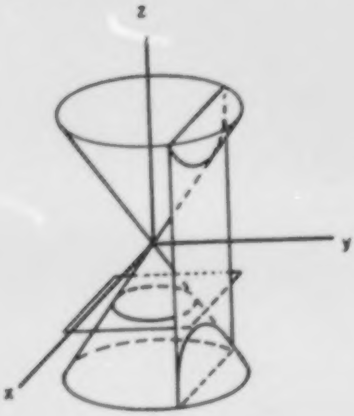


Figure 9-15

We find that the sections parallel to the xy-plane are circles whose radii increase as $|z|$ increases. The sections parallel to the other coordinate planes are hyperbolas. Does this sound familiar? It should, since the surface (Figure 9-15) is a right circular cone, whose sections are the conics we studied in Chapter 7.

A conical surface, or cone, is the surface generated by a line (called an element or generator) which moves so that it always contains a point of a plane curve (called the directrix) and a fixed point (called the vertex) which is not in the plane of the curve. (See Supplement to Chapter 7 for further information on the right circular cone and its sections.) Here we shall consider only right cones with vertex at the origin and the directing curve a conic section in a plane perpendicular to one of the coordinate axes.

As another example, let us sketch the graph of the equation

$$\frac{x^2}{4} - \frac{y^2}{1} + \frac{z^2}{9} = 0 . \tag{2}$$

The sections parallel to the xz-plane are ellipses; the cone (Figure 9-16) is called an elliptic cone.

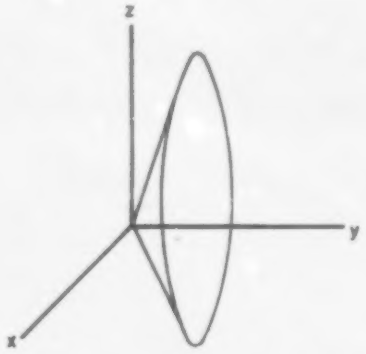


Figure 9-16

Exercises 9-5

Sketch the cones represented by Equations 1 to 6. On each sketch show the intercepts, traces, and at least two of the sections perpendicular to the axis of the cone.

1. $x^2 - z^2 = y^2$
2. $y^2 - z^2 = x^2$
3. $\frac{x^2}{4} - \frac{y^2}{25} + \frac{z^2}{4} = 0$
4. $\frac{x^2}{9} - \frac{y^2}{16} - \frac{z^2}{16} = 0$
5. $4x^2 + 9y^2 - 36z^2 = 0$
6. $16x^2 - 4y^2 + 9z^2 = 0$

Write an equation of each of the cones described in Exercises 7 to 10.

7. Axis is the y-axis, a perpendicular section is a circle whose radius is twice the distance from the origin to the plane of the section.
8. Axis is the x-axis, a perpendicular section at $x = 3$ is an ellipse whose section in that plane is $4y^2 + 9z^2 = 36$.
9. Axis is the z-axis, a perpendicular section at $z = 4$ is a circle of radius 3 .
10. Axis is the y-axis, a perpendicular section at $y = 5$ is an ellipse whose equation in that plane is $9x^2 + z^2 = 16$.

11. It was noted that the sections of the graph of Equation (2) parallel to the xz-plane are ellipses; prove that these ellipses all have the same eccentricity.

Challenge Problems

1. Write an equation of a cone whose axis is the x-axis, and whose sections perpendicular to the axis are ellipses with eccentricity $\frac{2}{3}$. At $x = 1$, the major axis of the ellipse is 12.
2. Write an equation of a cone whose axis is the z-axis, and whose sections perpendicular to the axis are ellipses with eccentricity $\frac{1}{2}$. At $z = 2$, the major axis of the ellipse is 16.

9-6. Surfaces of Revolution.

A surface that is generated by revolving a plane curve about a fixed line in the plane is called a surface of revolution. The fixed line is called the axis of the surface. Some of the quadric surfaces we have discussed here are surfaces of revolution. A sphere is one; it may be generated by revolving any of its great circles about a diameter of that circle. The ellipsoid of Figure 9-8, the paraboloid of Figure 9-9, the cylinder of Figure 9-12, and the cone of Figure 9-15 are all surfaces of revolution.

Let us find the equation of the surface obtained by revolving the parabola with equations $z^2 = 2y$, $x = 0$ about the y-axis. Let $P = (x,y,z)$ be a point on the surface. The plane through P perpendicular to the y-axis intersects the generating curve at the point $C = (0,y,k)$, where $k = d(C,F)$; the same plane intersects the y-axis at the point $F = (0,y,0)$. Since P must lie in this plane on a circle with F as center, its coordinates must satisfy the equation

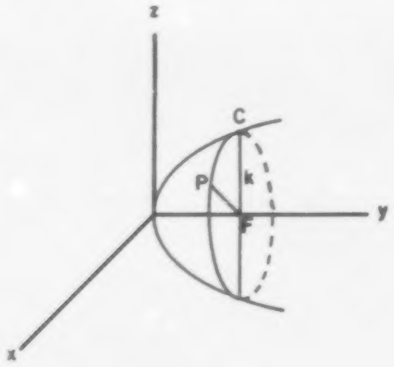


Figure 9-17

$$x^2 + z^2 = k^2, \tag{1}$$

where k is the radius of the circle. The value of k is determined by the requirement that $C = (0,y,k)$ be on the generating curve $z^2 = 2y$. Therefore,

$$(2) \qquad k^2 = 2y .$$

Equating the expressions for k^2 in Equations (1) and (2), we have

$$(3) \qquad x^2 + z^2 = 2y ,$$

an equation for the surface of revolution. It is, of course, a paraboloid.

The paraboloid of revolution for which we have just found an equation is generated by a parabola revolving on its axis. The parabola may revolve about lines other than its own axis; suppose it revolves about the z-axis. We sense intuitively that the resulting surface of revolution is quite different. Let us obtain its equation.

We start with equations of the generating curve,

$$z^2 = 2y , \; x = 0 ,$$

and let $P = (x,y,z)$ be a point on the surface. A plane through P perpendicular to the z-axis intersects the generating curve in $C = (0,k,z)$ where $k = d(C,F)$; the same plane intersects the z-axis in $F = (0,0,z)$.

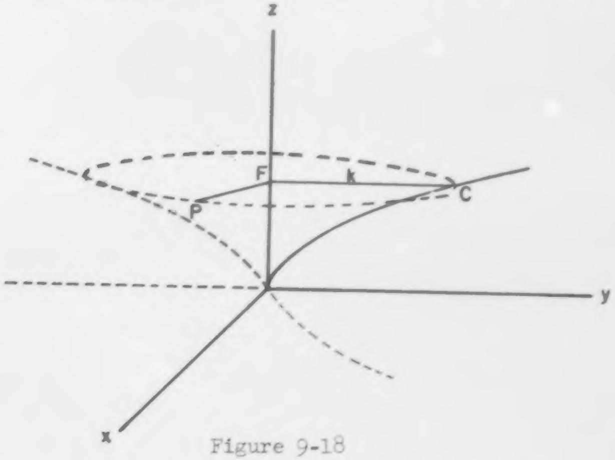


Figure 9-18

Since P lies on a circle in this plane with center F , its coordinates satisfy the equation

$$(4) \qquad x^2 + y^2 = k^2 .$$

Since k is the y-coordinate of C , and C is a point of the generating curve, the coordinates of C must satisfy the equation of that curve; hence

$$z^2 = 2k ,$$

and therefore

$$\frac{z^4}{4} = k^2 . \tag{5}$$

Equating the expressions for k^2 in Equations (3) and (4), we have

$$x^2 + y^2 = \frac{z^4}{4} \tag{6}$$

as an equation of our surface of revolution.

Since Equation (6) is not quadratic, the surface is not a quadric surface. However, we can use the methods of this chapter to investigate its shape. From the equation we see that the surface is symmetric with respect to each of the coordinate planes. Its only intersection with the xy-plane is the origin; the traces in the other coordinate planes are parabolas. The sections parallel to the xy-plane have equations of the form

$$x^2 + y^2 = \frac{k^4}{4} , \ z = k ;$$

clearly they are circles, as they should be for a surface of revolution.

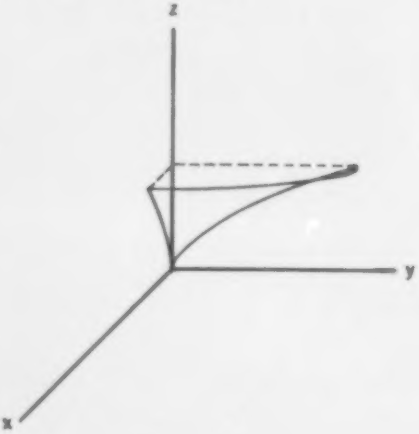


Figure 9-19

Exercises 9-6

In each of Exercises 1 to 18, find an equation of the surface obtained by revolving the plane curve about the axis indicated. Sketch the surface. In Exercises 1 to 10 the curve is to be revolved about its own axis, and the surfaces obtained are quadric surfaces; in Exercises 11 to 18 the axis of revolution is not an axis of the curve.

1. $z^2 = 8y$, $x = 0$; y-axis
2. $x^2 = 2z$, $y = 0$; z-axis
3. $3x = 2y$, $z = 0$; y-axis
4. $3x = 2y$, $z = 0$; x - axis
5. $y^2 + z^2 = 25$, $x = 0$; y-axis
6. $y^2 + z^2 = 25$, $x = 0$; z-axis

7. $9x^2 + 4y^2 = 36$, $z = 0$; x-axis

8. $9x^2 + 4y^2 = 36$, $z = 0$; y-axis

9. $4y^2 - z^2 = 16$, $x = 0$; y-axis

10. $x^2 - 4z^2 = 100$, $y = 0$; x-axis

11. $z^2 = 2x$, $y = 0$; z-axis

12. $x^2 = 2z$, $y = 0$; x-axis

13. $4y^2 - z^2 = 16$, $x = 0$; z-axis

14. $x^2 - 4z^2 = 100$, $y = 0$; z-axis

15. $y^2 = 8z$, $x = 0$; y-axis

16. $36y^2 - 4z^2 = 144$, $x = 0$; z-axis

17. $z = y^3$, $x = 0$; z-axis

18. $z = y^3$, $x = 0$; y-axis

19. If a curve in the yz-plane is represented by the equations $f(y,z) = 0$ and $x = 0$, show that, if $z \geq 0$, an equation of the surface obtained by revolving this curve about the y-axis is

$$f(y, \sqrt{x^2 + z^2}) = 0 .$$

9-7. Intersection of Surfaces. Space Curves.

In order to visualize quadric surfaces we have been discussing the intersections of curved surfaces and planes. This situation is represented by the simultaneous solution of two equations, such as

$$\begin{aligned} x^2 + y^2 + z^2 &= 25 , \\ z &= 3 . \end{aligned} \tag{1}$$

In this case, by substituting $z = 3$ into the first equation, we have $x^2 + y^2 = 16$, an equation of the circular section of the sphere in the plane $z = 3$. This circle is in a plane parallel to the xy-plane, has its center at $(0,0,3)$, and has radius 4 . It is completely described either by the first pair of equations or, more simply, by the pair

$$\begin{aligned} x^2 + y^2 &= 16 \\ z &= 3 . \end{aligned} \tag{2}$$

But Equations (2) represent the intersection of a cylinder and a plane. Or we might have

$$\begin{gathered} \frac{x^2}{16} + \frac{y^2}{16} - \frac{z^2}{9} = 0 \\ x^2 + y^2 = 16 , \end{gathered} \tag{3}$$

representing the intersection of a cone and a cylinder. In each case the circle which is the intersection of the two surfaces is the same. You might like to verify this by finding simultaneous solutions. (Equations (3) have an additional solution set.)

It should be intuitively evident by now that there are many pairs of surfaces which intersect in the circle described above. Earlier in your mathematical training you encountered this situation when you described a line as the intersection of two planes. There are infinitely many planes containing a given line, and any two of these planes may be used to describe the line. Similarly, there are infinitely many surfaces passing through a given curve, and this curve may be represented by the equations of any two of the surfaces having this curve as their intersection. Such an intersection is called a space curve. (It is perfectly correct to describe a plane as a surface and a line as a curve.)

From the many representations of a space curve, we try to choose one which gives us immediate information about the shape and location of the curve. For example, Equations (1) tell us at once that the intersection of their graphs is a circle and lies in the plane $z = 3$, but they do not show us the radius or the location of the center of the circle. Equations (3) indicate that the intersection of their graphs is a circle of radius 4, with its center on the z-axis, but we do not immediately see the plane of the circle. All of this information is available at first glance from Equations (2); hence, this representation is likely to be our choice from among the three suggested.

The representation of Equations (2) is useful also in sketching this space curve. Recall that by eliminating the variable z from $x^2 + y^2 + z^2 = 25$, we obtained the equation

$$x^2 + y^2 = 16 , \tag{4}$$

which represents a cylinder whose generators are parallel to the axis of the missing variable, z. Such a cylinder not only contains the curve, but its equation is also the equation of the projection of the curve on the coordinate plane. For this reason, this cylinder is sometimes called a projecting cylinder of the space curve. If the other variables are removed, other projecting cylinders are obtained; since these cylinders contain the curve, any two may be used to show the intersection. Interpreting Equations (2) in this way, we think of the plane $z = 3$ as a cylinder parallel to both the x-axis and the y-axis. For the sketch, we draw the projecting cylinder $x^2 + y^2 = 16$ and show the plane $z = 3$ intersecting it (Figure 9-20).

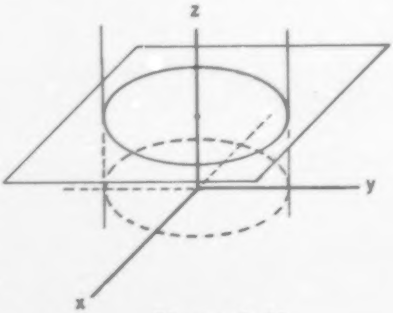


Figure 9-20

Example 1. Find simpler equations for the curve

$$\frac{x^2}{27} + \frac{y^2}{9} + \frac{z^2}{3} = 1 ,$$
$$x = 3 .$$

Solution. Let $x = 3$ in the first equation to obtain

$$\frac{9}{27} + \frac{y^2}{9} + \frac{z^2}{3} = 1 ,$$

or

$$\frac{y^2}{9} + \frac{z^2}{3} = \frac{2}{3} ,$$

which becomes

$$\frac{y^2}{6} + \frac{z^2}{2} = 1 .$$

The curve is an ellipse represented by

$$\frac{y^2}{6} + \frac{z^2}{2} = 1 ,$$
$$x = 3 .$$

Example 2. A typical problem from calculus could be stated as follows: Find the volume of the region in the first octant bounded by the surfaces $y^2 + z^2 + 2x = 16$, $x + y = 4$, and the coordinate planes.

As a start on this problem, you should make a reasonably accurate sketch of the boundaries of the region. (You can find the volume when you study calculus.) We first find the traces of the surfaces. One surface is a paraboloid of revolution and the other is a plane. Their traces are shown in Figure 9-21. These traces, along with the coordinate axes, provide us with all of the edges of the solid except one. This edge is the space curve which is the intersection of the paraboloid and the plane $x + y = 4$. To find this edge, we eliminate x from the equation of the paraboloid and obtain

$$(y - 1)^2 + z^2 = 9 ,$$

the projecting cylinder parallel to the x-axis. The projection on the yz-plane is a circle with center at $(0,1,0)$ and radius 3 , as is shown in the figure. The space curve is represented by

$$(5)\qquad \begin{aligned} (y-1)^2 + z^2 &= 9\ , \\ x + y &= 4\ , \end{aligned}$$

and we shall now describe how to locate some points on it.

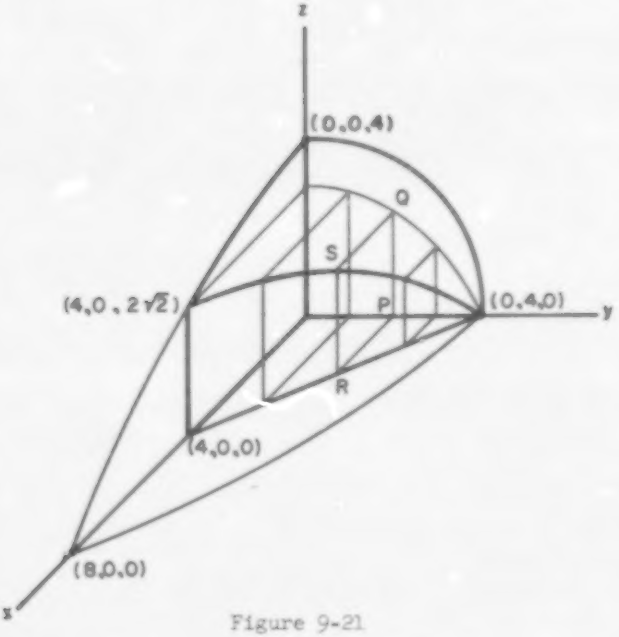


Figure 9-21

Since y is the variable appearing in both equations, we choose a point, P, on the y-axis, and we draw lines parallel to the other axes intersecting the traces of Equations (5) in points Q and R, as shown. We now complete the rectangle by drawing lines parallel to the x- and z-axes from Q and R. These lines intersect at S, a point of the space curve. Other points may be found in a similar manner, and when these points are joined by a smooth curve, the figure is completed.

Example 3. Sketch the curve described by

$$\begin{aligned} x &= 2\cos t\ , \\ y &= 2\sin t\ , \\ z &= 2t\ . \end{aligned}$$

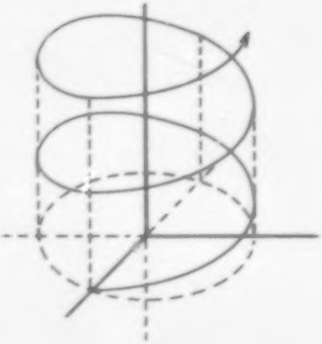


Figure 9-22

Solution. If we square both members of the first two equations and add, we obtain

$$x^2 + y^2 = 4(\cos^2 t + \sin^2 t)$$

or

$$x^2 + y^2 = 4 .$$

This represents a circular projecting cylinder of radius 2 whose axis is the z-axis. All elements of the solution set are contained in this cylinder, and since z is directly proportional to t , we note in Figure 9-22 that the curve is an ascending spiral "wrapping around" the cylindrical surface. This curve is called a helix.

We might view this differently by eliminating the parameter t . Then we have

$$x = 2 \cos \frac{z}{2}$$

$$y = 2 \sin \frac{z}{2} ,$$

and the curve is seen to be the intersection of two projecting cylinders whose cross-sections are sine (or cosine) curves. The elements of one cylinder are parallel to the y-axis; the elements of the other cylinder are parallel to the x-axis. If you wish to build a model for this problem, you might use two pieces of corrugated cardboard.

Still another view of this curve may be obtained by writing the equations in cylindrical coordinates. We shall consider this in the next section.

Exercises 9-7

1. Name and describe the intersection of each of the following pairs of equations, and write for each a simpler pair (if there is one).

   (a) $x^2 + y^2 + z^2 = 16$,
   $y = -2$.

   (b) $x^2 + y^2 + z^2 = 4$,
   $x = 3$.

(c) $x^2 + y^2 = 4$,
$z = 0$.

(d) $x^2 + y^2 + z^2 = 4$,
$z = 0$.

(e) $x^2 + z^2 = 25$,
$y = 5$.

(f) $x^2 + z^2 = 25$,
$z = 0$.

(g) $x^2 + y^2 = 50$,
$x - y = 0$.

(h) $x^2 + 8y^2 - 4z^2 = 12$,
$z = 1$.

(i) $x^2 + 3y^2 - 4z^2 = 12$,
$x = 0$.

(j) $x^2 + 2y^2 + 8z^2 = 8$,
$x = 0$.

(k) $x^2 + 2y^2 + 8z^2 = 8$,
$y = 2$.

(ℓ) $x^2 + y^2 - z^2 = z$,
$x^2 + y^2 - z^2 = 1$.

2. Make a sketch of the region in the first octant bounded by the given surfaces and the coordinate planes.

(a) Inside the cylinder $x^2 + y^2 = 50$ and under the plane $x + y + z = 10$.

(b) Inside the cylinder $y^2 + z^2 = 16$ and in the half-space formed by $x + y = 6$ which contains the origin.

(c) Inside the paraboloid $x^2 + y^2 = 4z$ and under the plane $z = 2$.

(d) Inside the cylinder $y^2 + z^2 = 25$ and inside the cylinder $x^2 + z^2 = 25$.

(e) Inside the sphere $x^2 + y^2 + z^2 = 25$ and inside the cylinder $y^2 + z^2 = 16$.

(f) Under the paraboloid $18z = 4x^2 + 9y^2$ and in the half-spaces formed by $x = 2$ and $y = 3$ which contain the origin.

3. Find the equations of the projecting cylinders of the curve whose equations are

$$x^2 + 2y^2 - z^2 = 3 ,$$
$$x^2 + y^2 - 2z^2 = -3 .$$

Sketch the curve by making use of the projecting cylinders.

*4. A calculus problem requires the student to find the height above the xy-plane in which the plane $2x + y = 2$ intersects the paraboloid $z = 16 - 4x^2 - y^2$. Find this height by sketching in one of the coordinate planes the trace of a projecting cylinder.

*5. A calculus problem asks for the volume inside the cylinder $x^2 + y^2 - 2y = 0$ and between the xy-plane and the upper nappe of the cone $z^2 = x^2 + y^2$. Make a sketch for this problem, showing the portion of the region in the first octant.

9-8. Cylindrical and Spherical Coordinate Systems.

Some problems in science that have a setting in 3-space are easier to handle if they are expressed in terms of cylindrical or spherical coordinates. If the surface has symmetry with respect to a line, then cylindrical coordinates may simplify the work of the problem. If the surface has point-symmetry, the use of spherical coordinates may provide a simpler analytic representation and solution.

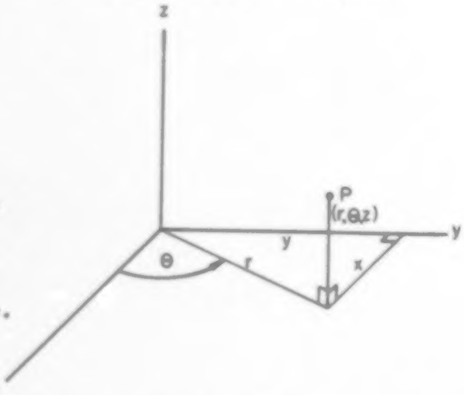


Figure 9-23

Cylindrical Coordinates are a combination of polar and rectangular coordinates. A polar coordinate system is used in one coordinate plane; the axis perpendicular to this plane has a linear coordinate system. A point is designated in cylindrical coordinates by an ordered triple. We use (r,θ,z), as indicated in Figure 9-23. The first two coordinates are the coordinates of the projection of P in the polar plane. The third coordinate is the coordinate of the projection of P on the linear axis. In this figure we may verify what we could have guessed; the transformations from cylindrical to rectangular form, and vice versa, are accomplished by the same process we used in Section 2-4 to relate polar and rectangular coordinates. The transforming equations are

$$\begin{array}{ll} x = r \cos \theta & r^2 = x^2 + y^2 \\ y = r \sin \theta & \tan \theta = \frac{y}{x}\ , \text{ where } \ x \neq 0 \\ z = z & z = z \end{array}$$

The simple equation, $r = k$, represents, in cylindrical coordinates, a right circular cylinder with radius k whose axis is the linear axis. This fact accounts for the name applied to this system.

Example 1. Write in cylindrical coordinates the equation of the sphere with radius $\sqrt{5}$ whose center is at the origin.

Solution. In rectangular coordinates the equation is $x^2 + y^2 + z^2 = 5$. Since $r^2 = x^2 + y^2$, the equation is written $r^2 + z^2 = 5$.

Example 2. Transform to rectangular coordinates and identify the surface whose equation in cylindrical coordinates is $3r \cos \theta + r \sin \theta + 2z = 0$.

Solution. Using the transforming equations, we obtain $3x + y + 2z = 0$, the equation of a plane.

Example 3. In connection with the helix in Example 3 of the previous section, we suggested a solution using cylindrical coordinates. We write θ in place of t, use the transforming equations, and square as before, obtaining

$$r^2 = x^2 + y^2 = 4 \cos^2 \theta + 4 \sin^2 \theta ,$$

$$r^2 = 4(\cos^2 \theta + \sin^2 \theta) ,$$

or

$$r^2 = 4 .$$

Since $r = 2$ has the same graph as $r^2 = 4$, we obtain a simple expression for the helix:

$$r = 2$$
$$r = 2\theta \ .$$

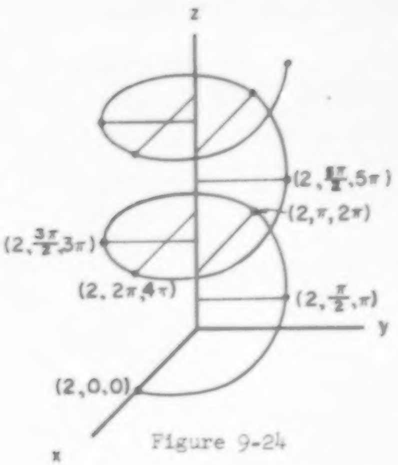

z
$(2, \frac{5\pi}{2}, 5\pi)$
$(2, \pi, 2\pi)$
$(2, \frac{3\pi}{2}, 3\pi)$
$(2, 2\pi, 4\pi)$
$(2, \frac{\pi}{2}, \pi)$
y
(2,0,0)
x

Figure 9-24

Since this helix is a constantly ascending spiral around the z-axis, we can locate some of its points by a device we might describe as fixing "ribs" to a "spine", or of locating steps on a spiral staircase. The z-axis will be the "spine" to which the "ribs" are attached. (We are using a condensed scale on the z-axis to save space.)

We first locate a point at (2,0,0) as shown in Figure 9-24. When $\theta = \frac{\pi}{2}$, we have rotated to a point one-quarter of the way around the "spine", and we have ascended a distance π . We fix a "rib" to this point. We might next stop at $\theta = \pi$ and fix another point. This process can be continued as long as desired and the points may be connected by a smooth curve to sketch a portion of the helix.

Another useful system for locating points in 3-space involves the use of spherical coordinates. In this system the coordinates of a point P are determined by assuming a polar coordinate system in the plane determined by the point P and the z-axis. The positive half of the z-axis is the polar axis and the positive sense of the polar angle is from the polar axis to ray $\overrightarrow{OP}$. The polar distances d(0,P) is denoted by ρ and the measure of the polar angle by ϕ . In the xy-plane the usual system of polar angles is assumed. The projection of P in the xy-plane determines the terminal side of a polar angle of measure θ . These three numbers represent the point P and are called the spherical coordinates of P . They are written as an ordered triple, usually as (ρ,θ,ϕ) . In Figure 9-25 this system is used to name the point which in rectangular coordinates would be P = (x,y,z) .

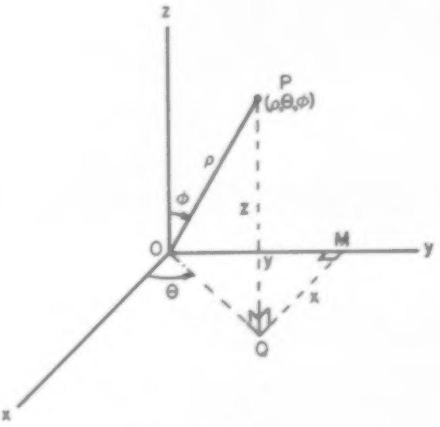


Figure 9-25

In order to relate spherical coordinates and rectangular coordinates, we obtain (from Figure 9-25) the following relations:

$$x = d(Q,M) = d(O,Q)\cos\theta = \rho\sin\phi\cos\theta ,$$
$$y = d(O,M) = d(O,Q)\sin\theta = \rho\sin\phi\sin\theta ,$$
$$z = \rho\cos\phi .$$

The de[illegible] the equations for relating spherical coordinates and cylindrical coordinates is left as an exercise.

Example 1. Write in spherical coordinates the equation of the sphere with radius $\sqrt{5}$ whose center is at the origin.

Solution. Since ρ is the distance from the origin to a point, we obtain

$$\rho = \sqrt{5} .$$

This simple equation form, $\rho = k$, for a sphere in spherical coordinates accounts for the name applied to this system. Compare this with $r = k$ in cylindrical coordinates and $r = k$ in polar coordinates.

Example 2. Transform to rectangular or cylindrical coordinates and identify the surface whose equation in spherical coordinates is $\rho\sin\phi = 3$.

Solution. We square both members and obtain

$$\rho^2\sin^2\phi = 9 .$$

Multiplying the left member by 1 (disguised as $\cos^2\theta + \sin^2\theta$), we have

$$\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = 9 ,$$
$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 9 ,$$

which in rectangular coordinates is

$$x^2 + y^2 = 9 .$$

In cylindrical coordinates we have simply

$$r = 3 .$$

This is the equation of a right circular cylinder with radius 3 whose axis is the z-axis.

It may come as a surprise when you realize that very likely you used spherical coordinates before you knew what they were. In terms of the position of a point on the earth, θ is the longitude, $90^\circ - \phi$ is the latitude, and (assuming the earth is a sphere) ρ is the earth's radius.

Exercises 9-8

1. Derive transforming equations to relate cylindrical coordinates and spherical coordinates.

2. Write the rectangular and the cylindrical coordinates of the points whose spherical coordinates are

   (a) $(4, \frac{\pi}{3}, \frac{\pi}{4})$.

   (b) $(3, 0, \frac{\pi}{3})$.

   (c) $(2, \frac{\pi}{2}, \frac{\pi}{2})$.

   (d) $(4, \frac{3}{2}, 1)$.

3. Write the rectangular and the spherical coordinates of the points whose cylindrical coordinates are

   (a) $(2, \frac{\pi}{6}, 3)$.

   (b) $(5, \frac{\pi}{2}, 0)$.

   (c) $(0, \frac{\pi}{4}, 8)$.

   (d) $(4, 1, 2)$.

4. Write the cylindrical and the spherical coordinates of the points whose rectangular coordinates are

   (a) $(2, 3, 0)$.
   (b) $(0, 6, 3)$.
   (c) $(2\sqrt{3}, 2, 4)$.
   (d) $(4, 1, 2)$.

5. Transform the following equations into cylindrical coordinates and into spherical coordinates.

   (a) $x^2 + y^2 = 25$.

   (b) $xz = 4y$.

   (c) $x^2 + y^2 = 8x$.

   (d) $x^2 + y^2 = 3z$.

6. Transform the following equations into rectangular coordinates.

   (a) $\rho = 6$.

   (b) $r = 6$.

   (c) $z = 6 + r$.

   (d) $z^2 = 9 - r^2$.

7. Identify and describe each of the following surfaces.

   (a) $r = 3$.

   (b) $\theta = \frac{\pi}{4}$.

   (c) $r^2 + z^2 = 4$.

   (d) $\phi = \frac{\pi}{4}$.

   (e) $\rho \cos \phi = 7$.

   (f) $z = r \cos \theta$.

   (g) $z = r$.

   (h) $r = 2 \sec \theta$.

8. A circular cylinder of diameter 4 intersects a sphere of radius 4 so that an element of the cylinder contains a diameter of the sphere. Choose axes and write equations of the bounding surfaces in

   (a) rectangular coordinates,

   (b) cylindrical coordinates, and

   (c) spherical coordinates.

9-9. Summary.

Our work in this chapter has been limited to the most important and familiar quadric surfaces, and we have located the coordinate axes so as to get simple equations for them. Students who have enjoyed this work may like to pursue it further by looking up such topics as ruled surfaces, hyperbolic paraboloids, curves in space, and surfaces of higher order.

Our objective here has been to develop methods to help you visualize surfaces and curves in space. The methods we have used are general, and should be of use to you in visualizing or sketching, particularly in your work in calculus and its applications.

Surfaces in space are represented by one equation, $f(x,y,z) = 0$; for quadric surfaces, the equation is of the second degree. Curves in space are given by the intersection of two equations (or three in parametric form), $f(x,y,z) = 0$ and $g(x,y,z) = 0$. The most important curves for sketching a surface are the traces and the sections parallel to the coordinate planes.

The surfaces we have studied include the cone, cylinder, sphere and ellipsoid, elliptic paraboloid, and the hyperboloid. A cone is generated by a line moving about a line with one point fixed, a cylinder by a line moving parallel to a fixed line, and a surface of revolution by a plane curve revolving about a line in the plane of the curve. For the limited cases we have studied, the quadric surfaces may be identified by their sections parallel to the coordinate planes as follows:

Quadric Surface	Sections Parallel to Coordinate Planes
Cone	Conic sections, including degenerate cases.
Elliptic or circular cylinder	Central ellipses or circles, parallel lines, or a line.
Sphere	Circles, including point-circle.
Ellipsoid	Ellipses, including circles and points.
Elliptic paraboloid	Parabolas and ellipses, including circles and points.
Hyperboloid	Ellipses, including circles and points, and hyperbolas

In sketching a surface, $f(x,y,z) = 0$, it is suggested that information about it be obtained and placed on the graph in the following order:

1. Intercepts — Set two of the variables equal to zero and solve the resulting equation for the third variable to find the intercepts on each axis.
2. Traces — Let the variables equal zero, one at a time, to find the equations of the traces - the sections in the coordinate planes.

3. Sections Let $z = k$, where k is a constant, to find the sections parallel to the xy-plane, for example. You can build up a sketch of the figure by using enough different values of k . For this purpose, select the sections easiest to draw.

We determine symmetry with respect to the xy-, yz-, or xy-plane by checking that the equation of the surface is unchanged when $-z$, $-x$, or $-y$ is substituted for z , x , or y , respectively. Knowing the symmetries of a surface helps in identifying it and sketching it. When a surface is symmetric, we often draw only the part in the first octant.

Certain curves which are the intersection of two surfaces were studied. In addition to using intercepts and traces, we used projecting cylinders to help us visualize and draw space curves.

Finally, cylindrical and spherical coordinates were introduced as other ways of describing the location of points in space.

Review Exercises

Discuss and sketch the surfaces represented by the equations in 1 to 20.

1. $16x^2 + 9y^2 + 16z^2 = 144$
2. $5x^2 + 5y^2 + 5z^2 - 45 = 0$
3. $16z = x^2 + y^2$
4. $36z = 9x^2 + 4y^2$
5. $25x^2 + 100y^2 = 400z$
6. $16x^2 + 9y^2 + 9z^2 = 144$
7. $9x^2 + 9y^2 + 9z^2 - 16 = 0$
8. $4x^2 - 9y^2 + 4z^2 = 36$
9. $4x^2 + 9z^2 = 36$
10. $4x^2 - 9z^2 = 36$
11. $9x^2 - 4y^2 = 0$
12. $36y^2 + 25z^2 = 900x$
13. $-16x^2 + 25y^2 + 16z^2 = 400$
14. $y^2 + z^2 = 100$
15. $x^2 + y^2 + z^2 - 2x - 3 = 0$
16. $25x^2 + 25y^2 + 25z^2 = 0$
17. $16x^2 - 9y^2 + 9z^2 = 0$
18. $x^2 + y^2 + z^2 + 8x - 6y + 10z + 34 = 0$
19. $36x^2 + 25z^2 = 900$
20. $25x^2 - 9y^2 - 9z^2 = 0$.

Discuss and sketch the surfaces described in Exercises 21 to 38. Write an equation for each surface; identify those that are not named.

21. A sphere centered at the origin with radius 10.

22. An ellipsoid with axes of lengths 12, 10, and 8.

23. A circular cylinder with radius 5 and axis the x-axis.

24. A prolate spheroid with axes of lengths 4 and 16.

25. An oblate spheroid with axes of lengths 4 and 6.

26. A cylinder with the y-axis as its axis, and its trace in the xz-plane the ellipse with equation $25x^2 + 16z^2 = 400$.

27. The surface obtained by revolving the curve with equations $16x^2 - 9y^2 = 144$, $z = 0$ about the y-axis.

28. The surface obtained by revolving the curve with equations $x^2 = 4z$, $y = 0$ about the z-axis.

29. The surface obtained by revolving the curve with equations $z^2 = 8y$, $x = 0$ about the y-axis.

30. The surface obtained by revolving the curve with equations $25x^2 - 36z^2 = 900$, $y = 0$ about the x-axis.

31. Refer to Exercise 27, but revolve about the x-axis.

32. Refer to Exercise 28, but revolve about the x-axis.

33. Refer to Exercise 29, but revolve about the z-axis.

34. Refer to Exercise 30, but revolve about the z-axis.

35. The surface obtained by revolving the curve with equations $25x^2 - 16y^2 = 0$, $z = 0$ about the x-axis.

36. Refer to Exercise 35, but revolve about the y-axis.

37. The surface obtained by revolving the line with equations $x = 2$, $y = 0$ about the z-axis.

38. Refer to Exercise 37, but revolve the line with equations $x = 2z$, $y = 0$.

39. Write an equation for the locus of points 10 units from $P = (3,-2,1)$.

40. Write an equation for the locus of points 5 units from the y-axis.

41. Write an equation for the locus of points equidistant from the plane $x = 0$ and the point $(6,0,0)$.

42. What are the graphs of the following equations?

(a) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

(b) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 0$

(c) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = -1$

(d) $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$

(e) $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0$

(f) $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = -1$

(g) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 1$

(h) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 0$

(i) $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = -1$

43. Points A and B are 4 units apart. Write an equation for the locus of a point the sum of whose distances from A and B is 6. Simplify the equation, sketch the graph, and identify it.

44. Follow the same instructions as in the previous exercise, but let the difference of the distances be 2.

45. A pencil with a hexagonal cross-section is sharpened. Describe the space curve which you see as the edge of the painted surface of the pencil.

46. A cube having edges 1 unit in length has one vertex at the origin and three of its faces each in one of the coordinate planes. A plane contains the midpoint of the diagonal of the cube from the origin and is perpendicular to the diagonal. Find the sections of this plane on the faces of the cube. What kind of figure is this set of sections?

47. Sketch the intersection of the surfaces

$$x^2 + y^2 + z^2 = 4 , \quad x^2 + y^2 - 4y = 0$$

in the first octant, using projecting cylinders.

48. In each of the following cases, classify the given surfaces, find the projecting cylinders of the curve of intersection, and sketch the curve.

(a) $x^2 + 2y^2 + z^2 = 8$, $3x^2 + 2y^2 - z^2 = 8$.

(b) $x^2 + 2y^2 + z^2 = 4$, $-2x^2 - y^2 + z^2 = 2$.

(c) $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + 2z^2 = 5$.

(d) $x^2 + y^2 = z$, $x^2 + y^2 = 4$.

49. Sketch the solid in the first octant bounded by the given surfaces and the coordinate planes.

(a) $x^2 + z^2 = 1$, $y = 2$.

(b) $y = x$, $z = x + y$, $x = 1$.

(c) $x^2 + y^2 = 9$, $z = y$, $z = 2y$.

(d) $x^2 + y^2 = 36$, $x^2 + z^2 = 25$.

50. Express each equation in terms of two other coordinate systems. (Assume that all relate to 3-space.)

(a) $z = 5$.

(b) $x^2 + y^2 = 4x$.

(c) $r = 7$.

(d) $x^2 + y^2 + z^2 = 25$.

(e) $r^2 + z^2 = 9$.

(f) $\rho \cos \phi = 6$.

(g) $x^2 - y^2 = 16$.

(h) $r = 2 \cos \theta$.

(i) $\rho \sin^2 \phi = 2 \cos \phi$.

(j) $\rho \sin \phi = 3$.

(k) $x^2 + y^2 = 64$.

(ℓ) $\rho \sin \theta \cos \phi = \cos \theta$.

Challenge Problems

Describe and sketch the surfaces represented by Equations 1 to 6 .

1. $z = \sin y$

2. $y = \cos x$

3. $z = x^2 - 2x$

4. $4x^2 + 9y^2 + 36z^2 + 8x - 54y - 72z = 23$

5. $x^2 + y^2 - 4z^2 + 2x + 6y + 8z = 10$

6. $z = \dfrac{x^2 - y^2}{x^2 + y^2}$

Chapter 10

GEOMETRIC TRANSFORMATIONS

10-1. Why Study Geometric Transformations?

In previous chapters you have had considerable experience in relating a graph and its analytic representation. Because of their importance, conic sections were given very careful treatment. Despite this emphasis you may have noticed that, with the exception of the circle, all the conics you sketched had their centers, foci, vertices at the origin and one or both of the coordinate axes as axes of symmetry.

However, in various studies where the graphs of the equations of conics (and other curves) are of importance, one encounters more complicated analytic representations of these curves. Consider, for example, the following pairs of equations:

(1) $x^2 + y^2 + 10x - 4y + 4 = 0$, $x^2 + y^2 = 25$;

(2) $x^2 - y^2 - 4x - 6y - 30 = 0$, $x^2 - y^2 = 25$;

(3) $y^2 - x - 6y + 11 = 0$, $y^2 = x$.

If you went to the trouble of graphing all six of these equations, you would find that each pair of equations represents a pair of congruent graphs. They differ only in their placement with respect to their coordinate axes. If one is interested in geometric properties of such graphs, it is clear that the second equation of each pair is simpler to analyze and will quite readily yield information regarding intercepts, symmetry, asymptotes, etc., relative to its coordinate system.

It is one of the purposes of this chapter to show how we can relate such a complicated equation of a curve to a simpler equation of the same curve represented in a different coordinate system. The operation which performs this task (among others) is commonly referred to as either a "transformation of axes" or a "transformation of coordinates".

In this chapter we will consider two types of transformations which accomplish the purpose just described. The type we treat first (in Sections 2 and 3) is one wherein the operation is performed on the axes and the graph under study remains fixed. We then turn our attention (in Sections 5 and 6) to the type wherein the operation is performed on the points of the curve while the axes remain fixed. We refer to the latter type as a point transformation.

Our task takes on one of two aspects. We may be given a relationship between the coordinates of $P = (x,y)$ on a curve C and the coordinates of $P' = (x',y')$ on a curve C' and then investigate the correspondence between C and C'. On the other hand, the converse is considered: Given two curves C and C' and some correspondence between them, we investigate the manner in which the coordinates of any point $P = (x,y)$ on C are related to the coordinates of the corresponding point $P' = (x',y')$ on C'.

In the cases of the three pairs of equations presented earlier, the corresponding curves were actually congruent and the point correspondence was one-to-one. In other cases the corresponding curves need not be congruent although there may still be significant relations between them. For example, in Section 6, you will encounter a correspondence between a straight line and a circle under a transformation called an inversion.

Certain transformations preserve geometric properties such as the measure of distance between points on the original curve, the measure of angle between two lines, the order of points on a line, etc. ... while others do not preserve these properties. Discovering which geometric properties are invariant (do not change) under a set of transformations is of significance to the advanced students of geometry because these properties help them to classify the large number of geometries which have been created. This topic is discussed in Section 4.

We may also speak of the properties of a transformation. An important transformation we shall meet in Section 6 has the property that it preserves measure of angle but not necessarily measure of distance. Transformations which have this property are called conformal and have many applications in science.

10-2. Translations.

Suppose we have a curve in the coordinate plane and an equation of the curve. Let us consider the problem of writing an equation of the same curve with respect to another pair of axes. The process of changing from one pair of axes to another is called "transformation of axes" or "a transformation of coordinates" as stated earlier.

One of the most useful, as well as simple, transformations is one in which the new axes are shifted in such a way that they remain parallel to their original positions and oriented in the same direction. Such a transformation is called a translation.

THEOREM 10-1. Given a coordinate system in a plane with origin at O. The axes are then translated so that the new origin is at $O' = (h,k)$. If (x,y) and (x',y') are the coordinates of a point P when referred to the original and new axes respectively, then $x' = x - h$ and $y' = y - k$.

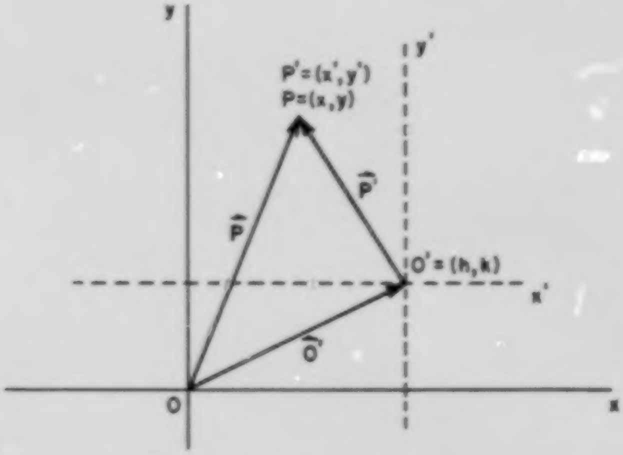


Figure 10-1

Proof. Let $\overrightarrow{P} = [x,y]$, $\overrightarrow{O'} = [h,k]$ and $\overrightarrow{P'} = [x',y']$.

(1) $\overrightarrow{P} = \overrightarrow{O'} + \overrightarrow{P'}$.

(2) $[x,y] = [h,k] + [x',y']$,
$= [h + x', k + y']$.

(3) Thus $\begin{cases} x = x' + h \\ y = y' + k \, . \end{cases}$ (Why?)

If we solve these equations for x' and y' , we obtain the "inverse form":

(4) $\begin{cases} x' = x - h \\ y' = y - k \, . \end{cases}$

We shall refer to the Equations (3) or (4) as the equations of translation.

Example 1. Find the new coordinates of the points $P_1 = (-3,1)$, $P_2 = (4,-2)$ if the origin is moved to $(-3,5)$.

Solution: Since $h = -3$, $k = 5$, the equations of translation are:

$$\begin{cases} x' = x + 3 \\ y' = y - 5 \, . \end{cases}$$

Applying these equations, we see that the point $P_1 = (-3,1)$ now has the coordinates $(0,-4)$, and $P_2 = (4,-2)$ now has the coordinates $(7,-7)$ with respect to the new axes.

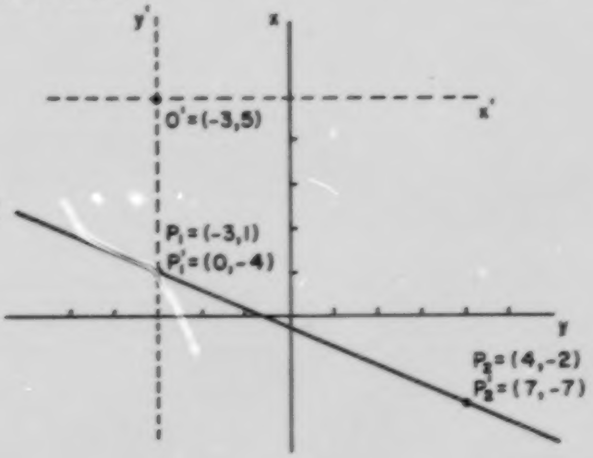


Figure 10-2

Consider an equation of a curve $f(x,y) = 0$. By the equations of translation, the coordinates x and y are transformed respectively into $x' + h$ and $y' + k$. Thus the equation $f(x,y) = 0$ changes to $f(x' + h, y' + k) = 0$. The two equations represent the same curve since the point $P(x,y)$ whose coordinates satisfy $f(x,y) = 0$ is the same as the point $P' = (x',y')$ whose coordinates satisfy $f(x' + h, y' + k) = 0$.

To illustrate this, consider the line ℓ in Figure 10-2 passing through the points P_1 and P_2 of Example 1. The equation of line ℓ is $3x + 7y + 2 = 0$. We now replace x by $x' - 3$ and y by $y' + 5$ and the equation of ℓ is now $3x' + 7y' + 28 = 0$. We note that the coordinates of points $P_1' = (0,-4)$ and $P_2' = (7,-7)$ satisfy this last equation. The new equation $3x' + 7y' + 28 = 0$ represents the *same* line, with respect to the new axes, x' and y', with the new origin at $O' = (-3,5)$.

Example 2. Find the equation of the circle $x^2 + y^2 + 10x - 4y + 4 = 0$ after a translation moves the origin to the point $(-5,2)$.

Solution: The equations of translation are $x = x' - 5$, $y = y' + 2$. Substituting into the equation of the circle, we have

$$(x' - 5)^2 + (y' + 2)^2 + 10(x' - 5) - 4(y' + 2) + 4 = 0 .$$

If we expand and collect terms, our equation simplifies to $x'^2 + y'^2 = 25$. We infer immediately that the circle has a radius of 5 units and that its center is at $O' = (-5,2)$. If you were to find the locus (or graph) of the original equation, you would discover that you had precisely the *same* circle. After doing this, you would appreciate the advisability of translating the axes. Note that the principal difference in the two equations is that one contains first degree terms and the other does not.

The basic question is: How do we know where to place the new origin so that a complicated equation reduces to a simple one? This method is illustrated in Example 3.

Example 3. Translate the axes so that the equation of the circle $x^2 + y^2 + 10x - 4y + 4 = 0$ can be written in a form which contains no first degree term.

Solution:

(1) Write the equation in the form $x^2 + 10x + y^2 - 4y = -4$ and complete the squares as follows:

$(x^2 + 10x + 25) + (y^2 - 4y + 4) = 4 + 25 - 4$ or

$(x + 5)^2 + (y - 2)^2 = 25$.

(2) If we let $x' = x + 5$ and $y' = y - 2$, our last equation becomes $x'^2 + y'^2 = 25$.

(3) We note that the equations $\begin{cases} x' = x + 5 \\ y' = y - 2 \end{cases}$ are the equations of translation to new axes with the origin at $(-5,2)$.

To show the wider applicability of this method, let us do one more example:

Example 4. Graph the curve $4x^2 - 9y^2 + 40x + 36y + 28 = 0$.

Solution.

(1) Rewrite the equation in the following form so that we can use the method of "completing the square":

$$4(x^2 + 10x) - 9(y^2 - 4y) = -28 .$$

(2) Completing the square:

$$4(x^2 + 10x + 25) - 9(y^2 - 4y + 4) = -28 + 100 - 36$$

or $$4(x + 5)^2 - 9(y - 2)^2 = 36 .$$

(3) Substituting $x' = x + 5$ and $y' = y - 2$, we have

$$4x'^2 - 9y'^2 = 36$$

or $$\frac{x'^2}{9} - \frac{y'^2}{4} = 1 .$$

We recognize this curve to be a hyperbola with center $O' = (-5,2)$. This curve can now be drawn by using the methods discussed in the earlier chapters.

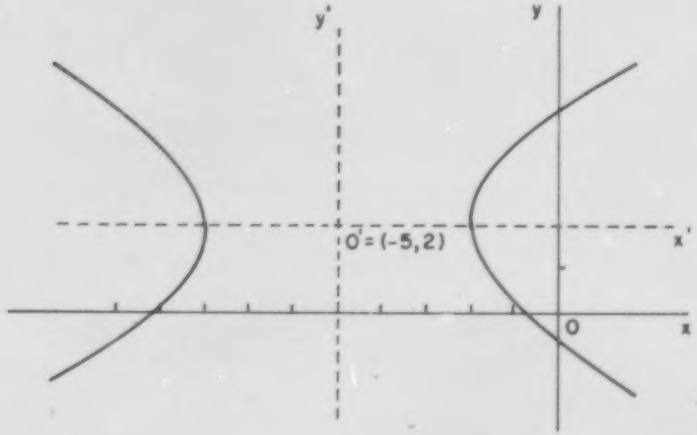


Figure 10-3

The translation of axes can be used to simplify equations of curves other than conics, but at this point we will restrict our discussions to such curves.

We will now generalize the above:

(1). A circle in the form $(x - h)^2 + (y - k)^2 = r^2$ can be simplified to $x'^2 + y'^2 = r^2$.

(2) An ellipse in the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ can be simplified to $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$.

(3) A hyperbola in the form $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ can be simplified to $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$.

(4) A parabola in the form $(y - k)^2 = 4p(x - h)$ or $(x - h)^2 = 4p(y - k)$ can be simplified to $y'^2 = 4px'$ or $x'^2 = 4py'$ respectively.

(5) The equilateral hyperbola $(x - h)(y - k) = c$ can be simplified to $x'y' = c$.

All of the above can be done by translating the axes to a new origin at $O' = (h,k)$ by use of the equations of translation

$$\begin{cases} x = x' + h \\ y = y' + k \,. \end{cases}$$

Exercises 10-2

1. Write the equations of translation which change the coordinates of $A = (2,12)$ to $(5,8)$ with respect to a new origin O'. What are the coordinates of O' with respect to the first origin?

2. Determine the equation of the curve represented by $2x^2 - y^2 - 12x - 4y + 12 = 0$ if the origin is translated to $(3,-2)$.

3. Given the transformation $\begin{cases} x' = x + 4 \\ y' = y + 6 \end{cases}$.

   What effect does this transformation have when it is applied to the curves:

   (a) $x^2 + y^2 = r^2$?

   (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

4. Points $A = (1,0)$, $B = (5,-2)$, and $C = (3,4)$ are vertices of a right triangle. Find the coordinates of these points if the origin is moved to $O' = (-4,-2)$ by a translation of axes. Using the new coordinates give two proofs that an observer at O' can present to demonstrate that $\triangle ABC$ is a right triangle.

5. Translate the axes so that the equation of the curve $x^2 - y^2 + 10x + 4y + 5 = 0$ can be written in a form containing no first degree terms. Indicate the equations of translation, draw both sets of axes, and sketch the curve.

6. Given circle $Q : x^2 + y^2 = 25$. Find the coordinates of three points A, B, and C on this circle. Then find their coordinates if the origin is translated to $O' = (1,-1)$ and the equation of the circle with respect to O'. Verify that the new coordinates of A, B, and C satisfy the transformed equation.

7. A line L has the equation $3x - 2y + 6 = 0$. Draw the line. The axes are then translated twice in succession in accordance with the equations

$$(1) \quad \begin{cases} x = x' + 3 \\ y = y' + 2 \end{cases} \quad \text{followed by} \quad (2) \quad \begin{cases} x' = x'' + 4 \\ y' = y'' + 5 . \end{cases}$$

Find the equation of L with respect to both the x'- and y'- and x''- and y''- axes. Then find the equations of translation which would perform both operations at once. What would be the effect of commuting translations (1) and (2) ?

8. Sketch the curves after performing a convenient translation of axes. Indicate the equations of translation and draw both sets of axes.

   (a) $y^2 - 6y - 12x - 3 = 0$

   (b) $3x^2 + 4y^2 - 6x + 8y - 5 = 0$

   (c) $2x^2 + 6x - 3y + 12 = 0$

   (d) $(x + 3)(y - 4) - 12 = 0$

   (e) $(y + 2)^2 = (x + 2)^3$

9. Derive the equations for the translation of axes with the new origin at $O' = (h,k)$ without the use of vectors.

10-3. Rotation of Axes: Rectangular Coordinates.

We next consider a rotation of a rectangular coordinate system C . We introduce a new coordinate system C' whose origin coincides with the origin of C and whose axes are obtained by rotating the axes of C through an angle α . Thus α is an angle in standard position whose initial side is the positive side of the x-axis and whose terminal side is the positive side of the x'-axis. Once again we want to discover the relationship between the coordinates of a point P in C and the coordinates of the same point in C'.

The presence of the angle α suggests the use of polar coordinates. We consider the systems of polar coordinates associated with C and C' by letting the polar axes be the positive sides of the x-axis and the x'-axis. Thus, as we have seen in Chapter 2 , if P is the point (r,θ) in the polar coordinate system whose polar axis is the positive side of the x-axis, then

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \tag{1}$$

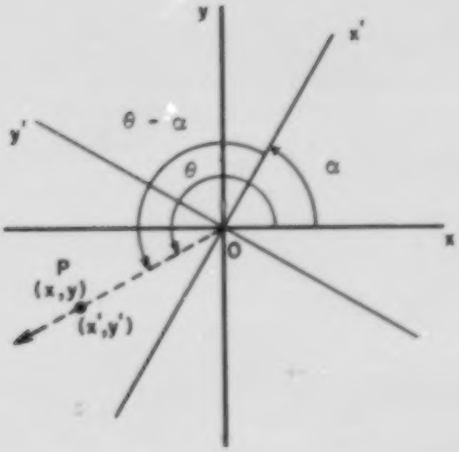


Figure 10-4

However, in the polar coordinate system whose polar axis is the positive side of the x'-axis, P is clearly the point $(r, \theta - \alpha)$. Therefore,

$$\begin{aligned} x' &= r \cos (\theta - \alpha) = r (\cos \theta \cos \alpha + \sin \theta \sin \alpha) \\ y' &= r \sin (\theta - \alpha) = r (\sin \theta \cos \alpha - \cos \theta \sin \alpha) \end{aligned} \tag{2}$$

Combining equations (1) and (2) , we get

$$\begin{aligned} x' &= x \cos \alpha + y \sin \alpha \\ y' &= -x \sin \alpha + y \cos \alpha \end{aligned} \tag{3}$$

These transformation equations are often called equations of rotation.

Example 1. In a given coordinate system, two points P_1 and P_2 have the coordinates $(2,3)$ and $(-4,5)$ respectively. The axes are then rotated through an angle of 30° . Find the rectangular coordinates of P_1 and P_2 with respect to the new axes.

Solution: Since $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$, we have upon substitution in the preceding equations,

$$\begin{cases} x' = \frac{1}{2}(\sqrt{3}x + y) \\ y' = \frac{1}{2}(-x + \sqrt{3}y) . \end{cases}$$

Thus P_1 has the new coordinates $\left(\frac{2\sqrt{3} + 3}{2}, \frac{-2 + 3\sqrt{3}}{2}\right)$;

P_2 has the new coordinates $\left(\frac{-4\sqrt{3} + 5}{2}, \frac{4 + 5\sqrt{3}}{2}\right)$.

Example 2. Find the equations relating coordinates in C and C' when C' is obtained from C by a rotation of (a) 45°, (b) -30°.

Solution:

(a) Since $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$, we have, upon substitution in the preceding equation,

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \frac{1}{\sqrt{2}}(x + y)$$

$$y' = \frac{-1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \frac{1}{\sqrt{2}}(-x + y)$$

(b) Since $\sin(-30^\circ) = -\frac{1}{2}$ and $\cos(-30^\circ) = \frac{\sqrt{3}}{2}$, we have

$$x' = \frac{\sqrt{3}}{2}x - \frac{1}{2}y$$

$$y' = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

We can solve for x and y in terms of x' and y' in Equation (3).

(1) $x \cos \alpha + y \sin \alpha = x'$

$-x \sin \alpha + y \cos \alpha = y'$.

(2) $x \cos^2 \alpha + y \sin \alpha \cos \alpha = x' \cos \alpha$

$x \sin^2 \alpha - y \sin \alpha \cos \alpha = -y' \sin \alpha$.

(3) Adding corresponding members, we have:

$$x \cos^2 \alpha + x \sin^2 \alpha = x' \cos \alpha - y' \sin \alpha \, ,$$

or $$x(\cos^2 \alpha + \sin^2 \alpha) = x' \cos \alpha - y' \sin \alpha \, ;$$

hence, $$x = x' \cos \alpha - y' \sin \alpha \, .$$

(4) By a similar process: $y = x' \sin \alpha + y' \cos \alpha$.

We shall refer to either of the pairs of equations

$$\begin{cases} x' = x \cos \alpha + y \sin \alpha \\ y' = -x \sin \alpha + y \cos \alpha \end{cases} \quad \text{or} \quad \begin{cases} x = x' \cos \alpha - y' \sin \alpha \\ y = x' \sin \alpha + y' \cos \alpha \end{cases}$$

as the equations of rotation.

Example 3. What equation represents the graph of $2x^2 + 4\sqrt{3}xy - 2y^2 = 16$ when the axes are rotated 30° ?

Solution

(1) Since $\theta = 30^\circ$, the equations of rotation are:

$$x = x' \cos \alpha - y' \sin \theta = \frac{1}{2}(\sqrt{3}x' - y')$$

$$y = x' \sin \alpha + y' \cos \theta = \frac{1}{2}(x' + \sqrt{3}y') \, .$$

(2) Substituting in the equation $2x^2 + 4\sqrt{3}xy - 2y^2 = 16$, and performing the indicated multiplications, we have

$$\frac{1}{2}(3x'^2 - 2\sqrt{3}x'y' + y'^2) + \sqrt{3}(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2)$$
$$- \frac{1}{2}(x'^2 + 2\sqrt{3}x'y' + 3y'^2) = 16 \, .$$

(3) Simplifying, we have $x'^2 - y'^2 = 4$.

We recognize the graph of this equation to be a hyperbola. The graph in the x'y'- coordinate system can easily be drawn.

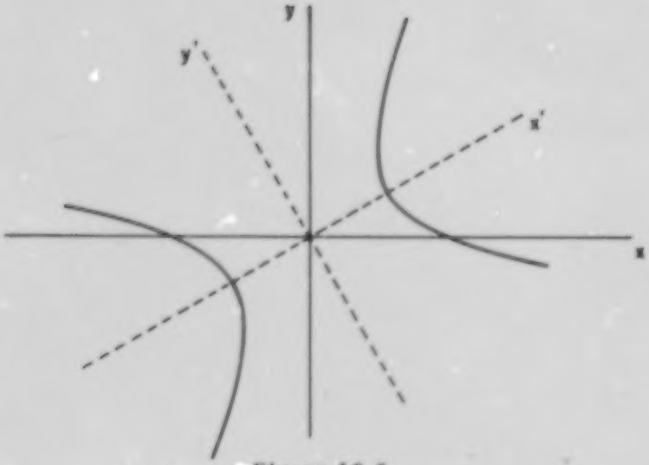


Figure 10-5

Note that a rotation of axes through an angle of 30° made the xy-term disappear. It was the elimination of the xy-term which made it possible for us to graph the curve much more readily. What we have not discussed is a method for determining through what angle a given set of axes may be rotated to eliminate the xy-term. Unfortunately we cannot develop this topic here. The interested student will enjoy studying this topic in the supplementary chapter.

Example 4. What equation represents the graph of $x^2 - y^2 = 4$ when the axes are rotated 45° ?

Solution

(1) Since $\alpha = 45^\circ$, the equations of rotation become:

$$x = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = \frac{1}{\sqrt{2}}(x' + y') .$$

(2) Substituting in the equation $x^2 - y^2 = 4$ we have

$$\frac{1}{2}(x'^2 - 2x'y' + y'^2) - \frac{1}{2}(x'^2 + 2x'y' + y'^2) = 4 .$$

(3) Simplifying, we have $x'y' = -2$.

We have here two different equations of the same equilateral hyperbola.

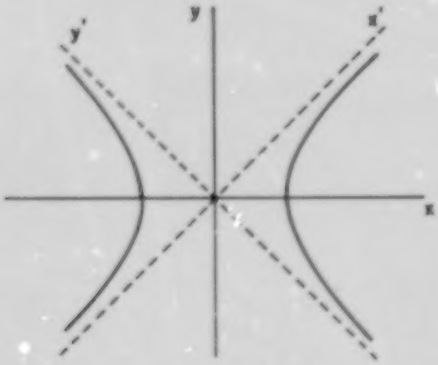


Figure 10-6

In this example, the equation with which we began had no xy-term. After a rotation, an xy-term appeared and the squared terms vanished. It may seem at first glance that we made a simple problem hard. There may be a good reason, however, why we may want to convert an equation from one form to another.

The equation $x'y' = -2$ tells us that the variables x' and y' are inversely proportional to each other. Inverse proportions are of frequent occurrence in science. For examples, in traveling a fixed distance at a constant rate the speed is inversely proportional to the time; the velocity of the wind is inversely proprotional to the spacing of the isobars (lines of constant pressure) on a weather map. We are trying to point out, in this instance, that the study of a curve whose equation has the form $xy = k$, a constant, may be more profitable than the study of the curve whose equation has the form $x^2 - y^2 = c$, a constant.

We now generalize the situation discussed in Example 4. If we start with a second degree equation containing no xy-term, a rotation of axes through an angle α, whose measure does not equal $\frac{k\pi}{2}$, for any integer k, will usually introduce an xy-term into the transformed equation. (An exception to this is the equation of a circle).

Consider the equation of the second degree which contains no xy-term.

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 ,$$

and apply the equations of rotation

$$x = x' \cos \alpha - y' \sin \alpha$$
$$y = x' \sin \alpha + y' \cos \alpha .$$

After we substitute and perform the indicated operations, this equation becomes:

$$A'x'^2 + B'x'y' + C'y'^2 + D'x + E'y + F' = 0$$

with respect to the new axes. The new constants are in terms of the constants A , C , D , E , and F . When $A' = C'$ and $B' = 0$ the equation represents a circle. (The details will be left as an exercise.)

This last equation is called the "General Equation of the Second Degree" and is written without primes as follows:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In the Supplement to Chapter 7, we consider the method of graphing such equations. In particular you will learn how to remove the xy-term by a rotation of axes through a determined angle θ . You have already learned how to translate the origin to remove the linear terms. When both of these operations are performed, the equation of the curve is in a form which is simpler to analyze and graph.

Polar Coordinates. It was pointed out earlier that when the polar axis is rotated through an angle whose measure is α , the point $P = (r,\theta)$ will have new coordinates $(r , \theta - \alpha)$. Figure 10-4 illustrated this relation.

Let us now consider a polar equation

$$(1) \qquad r = \frac{ep}{1 - e \cos (\theta - \alpha)}$$

which represents a conic whose axis makes an angle whose measure is α with the polar axis. We illustrate an ellipse in such a position in Figure 10-7.

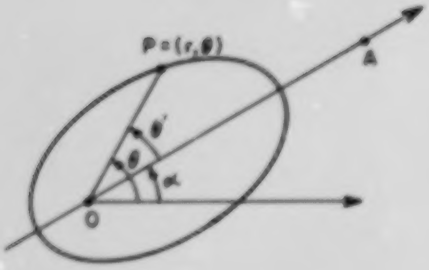


Figure 10-7

If the polar axis is now rotated through an angle whose measure is α, then an equation relative to the new polar axis, $\overrightarrow{OA}$, will be

$$(2) \qquad r = \frac{ep}{1 - e\cos\theta'}, \text{ where } \theta' = \theta - \alpha .$$

You will recognize this as a polar equation of a conic with focus at the pole and axis along the new polar axis as discussed in Chapter 7.

This rotation enables us to graph the same curve by using a simpler equation. This effect was observed earlier in Section 10-3 which was concerned with rectangular coordinates.

The polar equation which represents a circle is $r = k$, a constant. This equation is independent of θ and is not changed by any change in θ.

Example. Graph $x = \dfrac{18}{3 - 2\cos(\theta + 60^\circ)}$

Solution. We first rotate the polar axis through an angle of -60°. The equation of the curve relative to the new polar axis will be

$$r = \frac{18}{3 - 2\cos\theta'} .$$

This equation represents an ellipse with its focus at the origin and with its major axis along the new polar axis as shown in Figure 10-8.

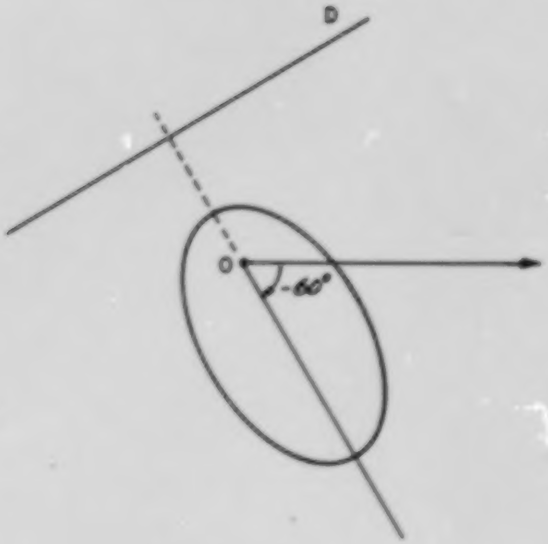


Figure 10-8

Exercises 10-3

1. Points $A = (1,0)$, $B = (5,-2)$, and $C = (3,4)$ are vertices of a right triangle. Find the coordinates of these points after the axes are rotated 150°. Using the new coordinates, show that the area of the triangle has not changed.

2. What is an equation in terms of x' and y' of the line $3x + 2y - 8 = 0$ after the axes have been rotated -30°? What is the slope of this line in the new coordinate system?

3. Given the equations of rotation

$$x = x' \cos \alpha - y' \sin \alpha ,$$
$$y = x' \sin \alpha + y' \cos \alpha .$$

Solve these equations for x' and y'.

4. What is an equation of the parabola $x^2 = y$ with respect to axes making an angle of 45° with the original axes?

5. Find the transformed equation if the axes are rotated through the indicated angle.

   (a) $x^2 - \sqrt{3}xy + 2y^2 = 3$, $\theta = 30^\circ$

   (b) $23x^2 + 8xy + 17y^2 = 25$, θ is the angle whose tangent equals $\frac{1}{2}$.

   (c) $xy = 4$. $\theta = \frac{3\pi}{4}$

   (d) $y^2 = 4x$, $\theta = \frac{\pi}{2}$

6. Given a circle whose equation is $x^2 + y^2 = r^2$. Find the equation of this circle with respect to the new axes after the original axes undergo a rotation through any angle whose measure is α .

7. Graph each of the following after rotating the polar axis to simplify the equation.

   (a) $r = \dfrac{6}{2 - \cos(\theta - 60^\circ)}$

   (b) $r = \dfrac{10}{5 + 3\cos(\theta - 120^\circ)}$

   (c) $r = \dfrac{3}{1 + \sin(\theta - 30^\circ)}$

Challenge Problems

1. Given the general equation of the second degree $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Find an equation of its graph if the axes are rotated through an angle of θ . Let A' , B' , and C' be the coefficients of x'^2 , $x'y'$, and y'^2 respectively. Prove that $B'^2 - 4A'C' = B^2 - 4AC$. (This expression $B^2 - 4AC$ is called the characteristic of the equation.)

2. A set of axes is rotated through an angle of measure α so that the equations of rotation are:

$$\begin{cases} x = x' \cos \alpha - y' \sin \alpha \\ y = x' \sin \alpha + y' \cos \alpha \end{cases}$$

This rotation is followed by a second rotation through an angle of measure θ so that the equations of rotation are:

$$\begin{cases} x' = x'' \cos \theta - y'' \sin \theta \\ y' = x'' \sin \theta + y'' \cos \theta \end{cases}$$

Prove analytically that the coordinates (x,y) and (x'',y'') are related by:

$$\begin{cases} x = x'' \cos (\theta + \alpha) - y'' \sin (\theta + \alpha) \\ y = x'' \sin (\theta + \alpha) + y'' \cos (\theta + \alpha) \end{cases}$$

10-4. Invariant Properties.

It was mentioned in Section 10-1 that certain properties of geometric objects often remain the same under transformations. Exactly which properties remain invariant depends, of course, upon the given transformations.

The geometry we are studying, called Euclidean geometry, is identified by the fact that the measure of both distance and angle of geometric figures remain invariant under translation and rotation of axes. Many other geometric properties also remain invariant. These include the order of points on a line, collinearity of points, and concurrence of lines. Here we shall discuss only the measures of distance and angle. The other geometric properties will be illustrated in the exercises.

We shall first consider the distance between two points in a plane under a translation of axes.

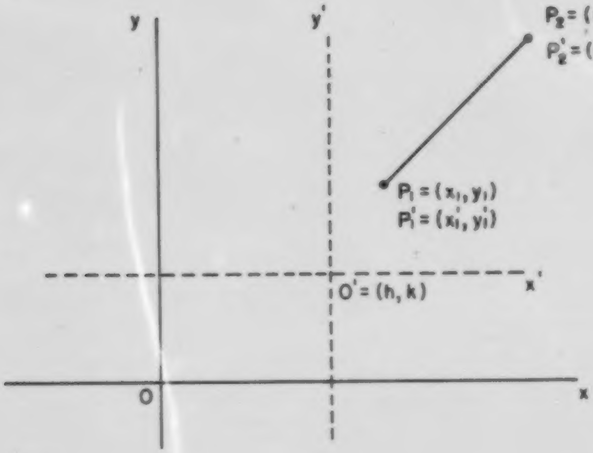


Figure 10-9

In the figure, the x-axis and y-axis with origin at O have been translated so that the new origin is at $O' = (h,k)$ with respect to the old axes. Observers at both O and O' look at the same two objects and consider the distance between them. The observer at O refers to their locations as positions P_1 and P_2, and the distance between them as s, while the observer at O' refers to the positions as P_1' and P_2' and the distance between them as s'.

You and I know that $s = s'$. But how can the two observers reconcile their observations? To answer this question, we list the known facts:

(1) $s = d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

and $s' = d(P_1',P_2') = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}$

(2) The equations of translation relating the coordinates are:

$$x' = x - h$$
$$y' = y - k$$

Using these facts, we have:

(3) $\begin{cases} x_2' = x_2 - h \\ x_1' = x_1 - h . \end{cases}$ Therefore $x_2' - x_1' = x_2 - x_1$,

and $\begin{cases} y_2' = y_2 - k \\ y_1' = y_1 - k \,. \end{cases}$ Therefore, $y_2' - y_1' = y_2 - y_1$

(4) We substitute the expressions from (3) in the formula for s', obtaining $s' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ which is identical with the formula for s, as was to be proved.

A numerical problem may help in making the above discussion clearer. Let $P_1 = (4,6)$, $P_2 = (-1,2)$ and $O' = (5,-2)$. Thus the equations of translation are $x' = x - 5$ and $y' = y + 2$.

The coordinates of P_1', are $(-1,8)$ and of P_2' are $(-6,4)$. Thus $d(P_1',P_2') = \sqrt{25 + 16} = \sqrt{41}$, $d(P_1,P_2) = \sqrt{25 + 16} = \sqrt{41}$, and we have $d(P_1,P_2) = d(P_1',P_2')$.

What if the axes in the above problem had been rotated instead of translated? We would then consider the following:

(1) The equations of rotation are:

$$\begin{cases} x' = x \cos\theta + y \sin\theta \,, \\ y' = -x \sin\theta + y \cos\theta \,. \end{cases}$$

so that $\begin{cases} x_1' = x_1 \cos\theta + y_1 \sin\theta \text{ and } x_2' = x_2 \cos\theta + y_2 \sin\theta \,; \\ y_1' = -x_1 \sin\theta + y_1 \cos\theta \text{ and } y_2' = -x_2 \sin\theta + y_2 \cos\theta \,. \end{cases}$

Therefore, $x_2' - x_1' = (x_2 - x_1)\cos\theta + (y_2 - y_1)\sin\theta$, and

$$y_2' - y_1' = -(x_2 - x_1)\sin\theta + (y_2 - y_1)\cos\theta \,.$$

(2) Squaring and adding corresponding members, we have:

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 = (x_2 - x_1)^2(\cos^2\theta + \sin^2\theta)$$
$$+ (y_2 - y_1)^2(\cos^2\theta + \sin^2\theta) \,,$$

or $(x_2' - x_1')^2 + (y_2' - y_1')^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

(3) Thus $d(P_1',P_2') = d(P_1,P_2)$.

We see that distance is invariant under both rotation and translation of axes and we state this as a theorem:

THEOREM 10-2. The measure of distance between two points is invariant under:

(a) a translation of axes

(b) a rotation of axes.

The invariance of the measure of angle under a translation or a rotation of axes follows directly from Theorem 10-2.

Consider $\triangle ABC$ determined by $A = (x_1,y_1)$, $B = (x_2,y_2)$, and $C = (x_3,y_3)$. Under either one of the above transformations, the points A , B , and C will have new coordinates. They will now be designated as $A' = (x_1',y_1')$, $B' = (x_2',y_2')$, and $C' = (x_3',y_3')$ with respect to the new axes.

Since distance between points is invariant, we have $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\overline{AC} \cong \overline{A'C'}$. Hence, $\triangle ABC \cong \triangle A'B'C'$ and the corresponding angles are congruent.

THEOREM 10-3. The measure of angle is invariant under:

(a) a translation of axes.

(b) a rotation of axes.

It would have been possible to prove the invariance of the measure of angle under translation or rotation independently of the invariance of distance discussed here. We could start with the formula

$$\cos\theta = \frac{1 + m_1 m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}} ,$$

and consider the lines $L_1 : ax + by + c = 0$ and $L_2 : a'x + b'y + c' = 0$.

Upon the translation of axes, the lines L_1 and L_2 , with respect to the new axes have the equations

$$L_1' : a(x' + h) + b(y' + k) + c = 0 ,$$

or

$$L_1' : ax' + by' + (ah + bk + c) = 0 .$$

and

$$L_2' : a'(x' + h) + b'(y' + k) + c' = 0 ,$$

$$L_2' : a'x' + b'y' + (a'h + h'k + c') = 0 .$$

The slope of L_1' is given by $m_1' = \frac{a}{b} = m_1$, and the slope of L_2' is given by $m_2' = -\frac{a'}{b'} = m_2$. Since the slopes are equal, $\cos \theta' = \cos \theta$ and $\theta' = \theta$ for the principal value. Hence the measure of angle is invariant under translation.

The proof of the invariance of angle under rotation involves considerable algebraic manipulation and is left as a "challenge" exercise.

Exercises 10-4

1. (a) Find an equation of the line through $A = (2,1)$ and $B = (0,4)$ and draw the line.
   (b) Find the coordinates of A and B and an equation of the line after the origin has been translated to $(-4,-6)$.
   (c) Verify that $d(A,B)$ is invariant under this translation.

2. (Refer to Exercise 1 above)
   (a) Find the coordinates of A and B and an equation of the line after the axes have been rotated 90° .
   (b) Verify that $d(A,B)$ is invariant under this rotation.

3. Given line $L : 4x - 3y - 12 = 0$ passing through $A = (0,-4)$, $B = (2,-\frac{4}{3})$ and $C = (3,0)$.
   (a) Find the coordinates of these points (now renamed A' , B' , and C' respectively) and an equation of the line (now called L') when the origin has been translated to $(-1,-1)$.
   (b) Verify that the order of points A' , B' , and C' is the same as that of A , B , and C . (That is, order of points on a line is invariant.)
   (c) Verify that A' , B' , and C' are collinear. (That is, collinearity of points is invariant under translation.)

4. Given lines $L_1 : 4x - 3y - 5 = 0$, $L_2 : x - 2y = 0$, and $L_3 : 5x - 3y - 7 = 0$.
   (a) Verify that L_1 , L_2 , and L_3 are concurrent.
   (b) Find equations of these lines (now renamed L_1' , L_2' and L_3') after the origin has been translated to $(3,-2)$.

(c) Verify that L_1', L_2' and L_3' are concurrent. (That is, concurrence of lines is invariant under translation.)

(d) What is the relation between the point of concurrency of L_1, L_2 and L_3 and that of L_1', L_2' and L_3'?

(e) Do parts (b), (c), (d) if, instead of translating the origin, the axes are rotated 45°.

5. Given lines $L_1 : 3x + 2y - 8 = 0$

and $L_2 : 5x - y - 9 = 0$.

(a) Find the acute angle between L_1 and L_2 at their point of intersection.

(b) Find equations of L_1 and L_2 (now called L_1' and L_2') after the origin is translated to $(2,2)$.

(c) Find the angle between L_1' and L_2' and verify that the angle is invariant under translation.

Challenge Problem

Prove that the measure of angle is invariant under a rotation of axes, without making use of the invariance of distance.

10-5. Point Transformations

In the previous sections we considered an operation called the "transformation of axes". We now consider another type of tranformation which achieves similar results from a different point of view. However, this new point of view leads to significant results, such as the transformation of a given curve into a corresponding curve which is not congruent to the original. This we could not achieve by the original approach.

We now consider a transformation, called a point transformation, which carries each point A into another point A' in the same plane. Thus the points of a figure F are carried into a set of points forming a figure F', as shown in Figure 10-10. The axes remain fixed.

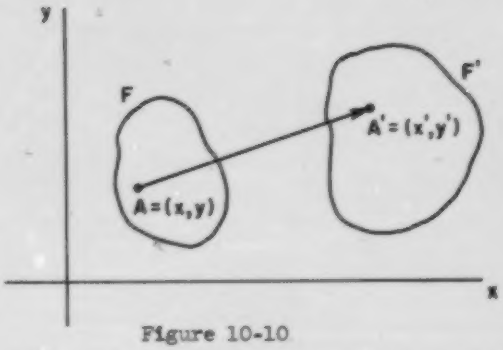


Figure 10-10

In this sense a transformation is an operation by which each element of a geometric figure is replaced by another element. Another way of expressing this concept is that a transformation is a one-to-one correspondence or mapping of each point of A onto a corresponding point A' . The plane is mapped onto itself. A point transformation is written symbolically as $A \longrightarrow A'$ and A' is called the image of A .

We can also consider translations and rotations as point transformations. In Figure 10-11, $P = (x,y)$ has been mapped into $P' = (x',y')$ by "moving" the point horizontally a distance of h and vertically a distance of k . Thus

$$\begin{cases} x' = x + h \\ y' = y + k \, . \end{cases}$$

Another way to write this transformation is $(x,y) \longrightarrow (x + h, y + k)$. This form will be used frequently in the remainder of the text.

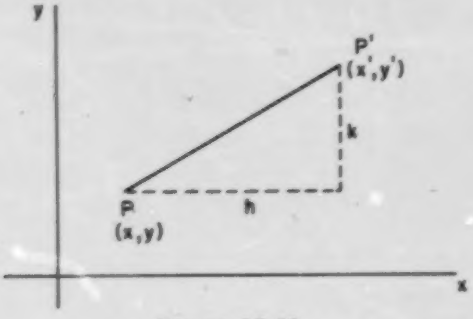


Figure 10-11

This pair of equations is similar to those derived earlier for a translation of axes; they differ only in the signs of h and k . This occurs because we are now moving the point and keeping the axes fixed.

The following example will illustrate this fact.

Let points $A = (2,0)$, $B = (2,1)$ and $C = (4,1)$ be the vertices of a triangle as shown in Figure 10-12. These points now undergo a point transformation given by

$$\begin{cases} x' = x + 4 \\ y' = y + 6 \, . \end{cases}$$

Thus

$$A = (2,0) \longrightarrow A' = (6,6)$$
$$B = (2,1) \longrightarrow B' = (6,7)$$
$$C = (4,1) \longrightarrow C' = (8,7)$$

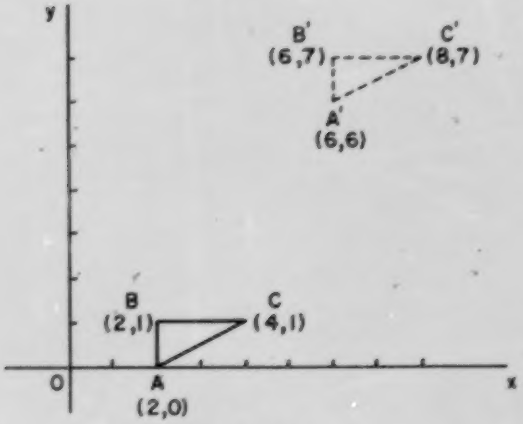


Figure 10-12

You will note that $\triangle ABC$ has been mapped into $\triangle A'B'C'$. You should also observe that the same "visual effect" could have been achieved by translating the x- and y-axes to a new origin at $(-4,-6)$. What we are saying is that $\triangle ABC$ would have the same relative position and appearance to a person standing at point $(0,0)$ as $\triangle A'B'C'$ would have to a person standing at point $(-4,-6)$. Note that the coordinates $(-4,-6)$ are the negatives of the values of h and k used in the point transformation.

A rotation is now considered as a mapping in which each point in the plane is mapped onto a point the same distance from the origin as previously. When $P \longrightarrow P'$ and $Q \longrightarrow Q'$, the rotation will map $\angle POP'$ into the congruent angle QOQ' . In the figure, $A = (2,0)$ has been mapped onto

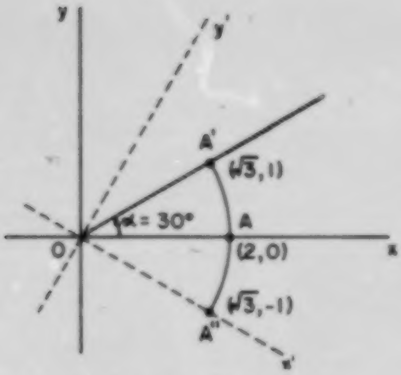


Figure 10-13

A' = $(\sqrt{3},1)$ by rotating through an angle whose measure is 30° ; both points are at a fixed distance of two units from O . A comparable visual effect would have been achieved if the axes had been rotated through an angle whose measure is -30° , and A" = $(\sqrt{3},-1)$ located on the x'-axis. The idea we are emphasizing is that A has the same relative position to an observer at A" as A' has to an observer at A . Also, $\overline{OA}$ has the same position with respect to the x'- and y'-axes as $\overline{OA'}$ has with respect to the x- and y-axes. A similar statement could be made regarding the rotation of any polygon or for any general figure F . The angle of rotation could be generalized to be any angle whose measure is α .

We now return to the concept of reflection which was discussed in detail in Section 6-2 with relation to the symmetry of curves. We shall now define certain relfections in terms of point transformation as follows:

(1) A reflection with respect to the x-axis is given by $(x,y) \rightarrow (x,-y)$

(2) A reflection with respect to the y-axis is given by $(x,y) \rightarrow (-x,y)$

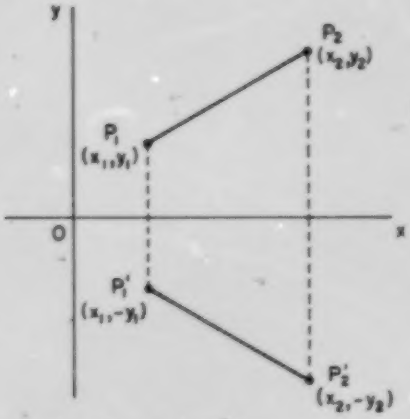
(3) A reflection with respect to the origin is given by $(x,y) \rightarrow (-x,-y)$.

Note our use here of the alternate notation indicated earlier in this section.

Reflections with respect to lines L and L' parallel to the x- and y-axes respectively are best treated by translating the x- and y-axes to coincide with L and L' . In accordance with our practice regarding notation we shall now refer to lines L and L' as the x'- and y'-axes respectively. Thus the point transformations are considered with respect to the x'- and y'-axes and to the new origin at O' = (h,k) as shown in Figure 10-14d.

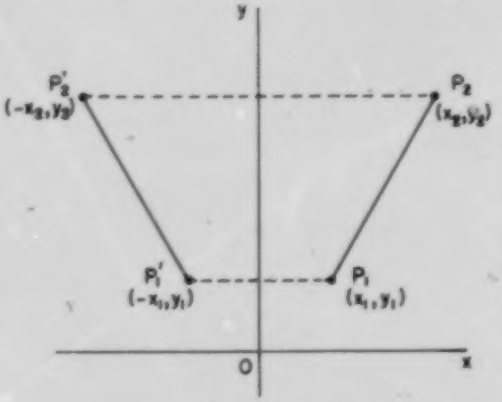
We can consider reflections with respect to any point or line but the equations of transformation are often difficult to state explicitly. We consider this subject beyond the scope of this text and refer you to the challenge exercises in Section 6-2.

Sone reflections of segments are indicated in Figure 10-14.


y
P_2
(x_2,y_2)
P_1
(x_1,y_1)
O
x
P_1'
$(x_1,-y_1)$
P_2'
$(x_2,-y_2)$

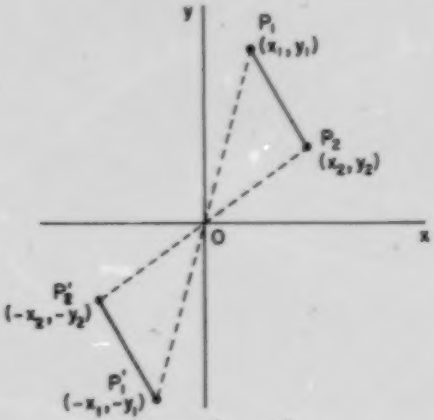
$(x,y) \longrightarrow (x,-y)$

Figure 10-14a


y
P_2'
$(-x_2,y_2)$
P_2
(x_2,y_2)
P_1'
$(-x_1,y_1)$
P_1
(x_1,y_1)
O
x

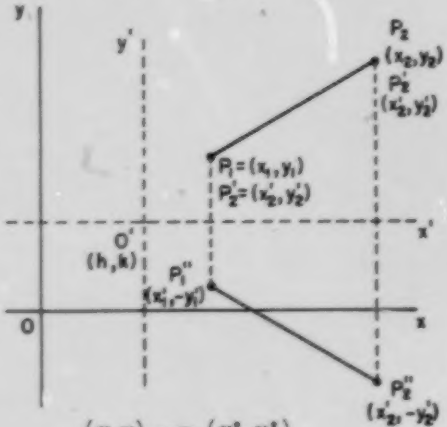
$(x,y) \longrightarrow (-x,y)$

Figure 10-14b


y
P_1
(x_1,y_1)
P_2
(x_2,y_2)
O
x
P_2'
$(-x_2,-y_2)$
P_1'
$(-x_1,-y_1)$

$(x,y) \longrightarrow (-x,-y)$

Figure 10-14c


y
y'
P_2
(x_2,y_2)
P_2'
(x_2',y_2')
$P_1=(x_1,y_1)$
$P_2'=(x_2',y_2')$
x'
O'
(h,k)
P_1''
$(x_1',-y_1')$
O
x
P_2''
$(x_2',-y_2')$

$(x,y) \longrightarrow (x',y')$

$(x',y) \longrightarrow (x',-y')$

Figure 10-14d

In each of the above illustrations, $d(P_1,P_2) = d(P_1',P_2')$. It is possible to prove that distance is invariant under the set of all reflections. We present here a proof of the first case where a line segment is reflected with respect to the x-axis.

Referring to Figure 10-14a, we have $d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and $d(P_1',P_2') = \sqrt{(x_2 - x_1)^2 + (-y_2 + y_1)^2}$. Since $(-y_2 + y_1)^2 = (y_2 - y_1)^2$ we have $d(P_1,P_2) = d(P_1',P_2')$.

It is also possible to prove that any translation, rotation, or combination of translations and rotations, can be accomplished by a series of no more than three line reflections. A proof will be found in the Supplement to Chapter 10. We shall merely illustrate it here in three examples.

Example 1. Show how the translation of $\triangle ABC$ to the new position indicated by $\triangle A''B''C''$ can be effected by a series of line reflections.

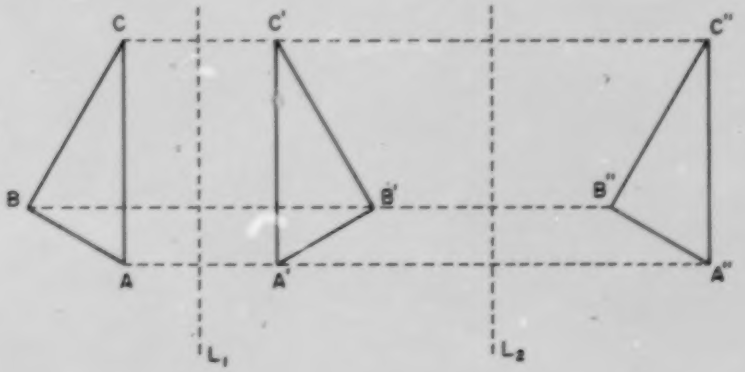


Figure 10-15

In Figure 10-15, we see that $\triangle ABC$ has been translated to $\triangle A''B''C''$ by a series of two refelctions. The axes of reflection, L_1 and L_2 , were selected parallel to $\overline{AC}$. Axis L_1 may be chosen freely but there is only one position possible for L_2 .

Example 2. (Same as Example 1.)

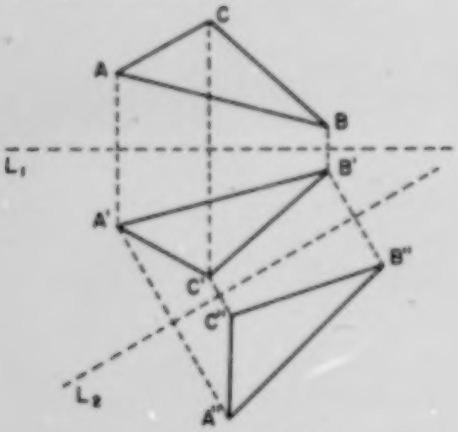


Figure 10-16

In Figure 10-16, we observe that $\triangle ABC$ has been reflected with respect to axes L_1 and L_2, with the result that it has been both translated and rotated.

Example 3. Demonstrate how axes of reflection can be selected to move a directed line segment from one position to another given position.

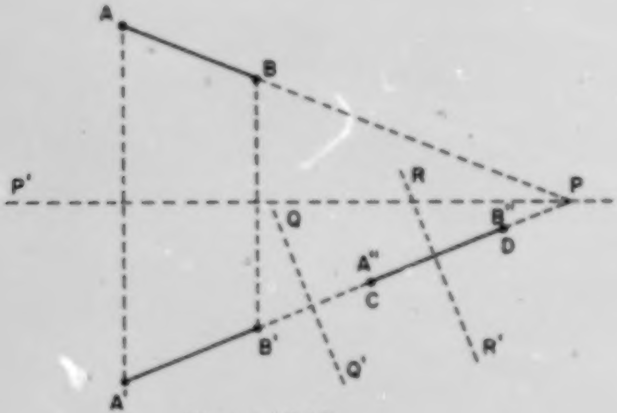


Figure 10-17

In Figure 10-17, $\overline{AB} \longrightarrow \overline{A''B''}$ by a series of at most three line reflections by using the following procedure.

(1) Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{A''B''}$ intersecting at P.

(2) Bisect angle P and call the bisector $\overleftrightarrow{PP'}$.

(3) Reflect $\overline{AB}$ with respect to $\overleftrightarrow{PP'}$. $\overline{A'B'}$, the image of $\overline{AB}$, will lie on $\overleftrightarrow{A''B''}$.

(4) Construct $\overleftrightarrow{QQ'}$, a perpendicular to $\overline{B''A''}$. Reflect $\overline{A'B'}$ with respect to $\overleftrightarrow{QQ'}$. Its image $\overline{DC}$ lies on $\overleftrightarrow{A'B'}$ and coincides with $B''A''$.

(5) Construct $\overleftrightarrow{RR'}$, the perpendicular bisector of $\overline{CD}$. Reflect $\overline{CD}$ with respect to $\overleftrightarrow{RR'}$. Thus $D \rightarrow A''$ and $C \rightarrow B''$ and the order of points on $\overline{A''B''}$ is the same as that of $\overline{AB}$.

The selection of axes of reflection when $\overline{AB} \| \overline{A''B''}$ is left as an exercise.

The effect of one or more reflections upon a geometric figure can be studied analytically as well as by actual construction and observation. To illustrate this approach, we shall consider the point reflection $(x,y) \rightarrow (-x,-y)$.

Upon applying this transformation to the line $L : ax + by + c = 0$, the equation becomes $L' : -ax - by + c = 0$ or $ax + by - c = 0$. The lines L and L' are parallel but the intercepts on the axes have different signs. Specifically, the line $2x + 3y - 6 = 0$, with intercepts $(3,0)$ and $(0,2)$ transforms to the line $2x + 3y + 6 = 0$ with intercepts $(-3,0)$ and $(0,-2)$.

When the same transformation is applied to the circle $x^2 + y^2 = r^2$, we note that there is no change in the equation. This result verifies the fact that this circle is symmetric with respect to the origin. A similar result is obtained for the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, the hyperbolas $b^2x^2 - a^2y^2 = a^2b^2$ and $xy = k$, the cubic parabola $y = x^3$, and any other curves that are symmetric to the origin.

The circle $x^2 + y^2 + Dx + Ey + F = 0$ transforms into another circle $x^2 + y^2 - Dx - Ey + F = 0$. The radii have the same measure but the center is now at $(\frac{D}{2}, \frac{E}{2})$ instead of at $(-\frac{D}{2}, -\frac{E}{2})$. Figure 10-16 illustrates the effect of the point reflection $(x,y) \rightarrow (x'y')$ upon the circle $C : x^2 + y^2 - 4x - 6y - 12 = 0$. The equation of the transformed circle is $C' : x^2 + y^2 + 4x + 6y - 12 = 0$. C and C' both have a radius of 5 but the center of C' is at $(-2,-3)$ while that of C is at $(2,3)$.

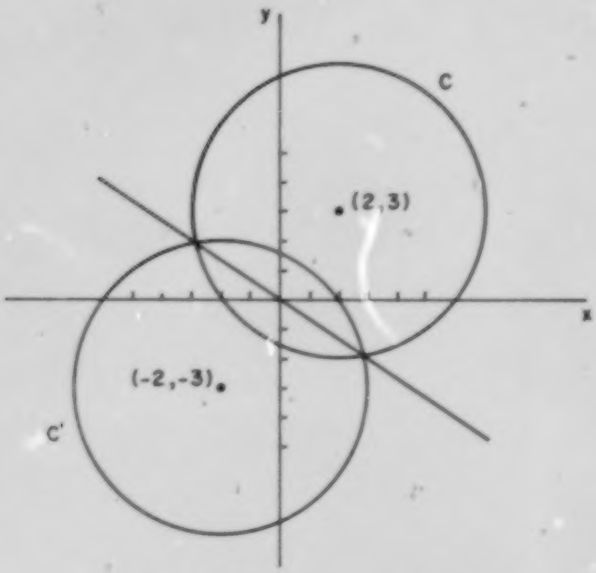


Figure 10-18

A second reflection $(x',y') \longrightarrow (x'',y'')$ with respect to the same point will map C' into $C'' : x^2 + y^2 - 4x - 6y - 12 = 0$ and we observe that $C'' = C$. A similar result is obtained when any reflection is followed by one of the same type and with respect to the same point or line. A number of transformations, other than reflections have this same property. We shall discuss one of these in the next section.

A variety of point transformations will be presented in the exercises.

Exercises 10-5

1. Given points $A = (1,2)$ and $B = (3,-4)$. Reflect A and B with respect to the

   (a) x-axis

   (b) y-axis

   (c) origin

   (d) line $x = 6$.

   Verify in each case that $d(A,B)$ is invariant.

2. The equation $x' = x + 2$ may represent a point transformation along the x-axis. Select any three points on the x-axis, find their images under the transformation, and determine two properties which remain invariant.

3. Perform Exercise 2 for the transformation $x' = 2x$. Find three properties invariant under this transformation.

4. Show that the angle between the lines $L_1 : y = 0$ and $L_2 : y = x$ is preserved under rotation through an angle of measure $\frac{\pi}{4}$.

5. Show the effect of the mapping indicated for each of the following curves by graphing both the original curve and its image on the same set of axes.

   (a) $y^2 = x$; $(x,y) \longrightarrow (-x,y)$

   (b) $x^2 = y$; $(x,y) \longrightarrow (-x,-y)$

   (c) $xy = 6$; $(x,y) \longrightarrow (-x,-y)$

   (d) $4x^2 - 9y^2 = 36$; $(x,y) \longrightarrow (3x,2y)$

   (e) $x^2 + y^2 - 2x + 4y + 4 = 0$; $(x,y) \longrightarrow (-x,y)$

   (f) $y = x^3$; $(x,y) \longrightarrow (x,-y)$

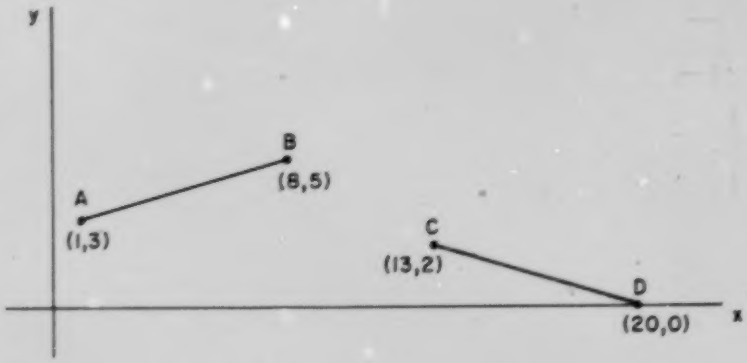
   (g) $y = \sin x$; $(x,y) \longrightarrow (x,-y)$

   (h) $y = \tan x$; $(x,y) \longrightarrow (-x,y)$

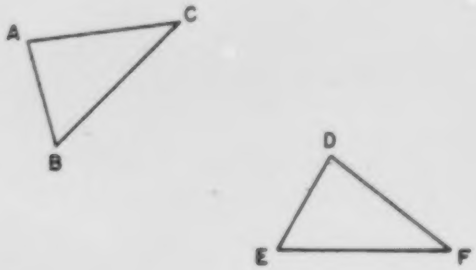
   (i) $y = 2^x$; $(x,y) \longrightarrow (-x,y)$

6. $A = (-2,1)$, $B = (5,-2)$, and $C = (3,3)$ are vertices of a triangle. They are rotated about the origin through an acute angle θ such that $\tan \theta = \frac{3}{4}$. Test and verify three properties which remain invariant under this rotation.

7. (a) Given the segments $\overline{AB}$ and $\overline{CD}$ as shown in the figure. Show, by construction, how $\overline{AB}$ can be mapped into $\overline{CD}$ by means of line reflections.



(b) Trace congruent triangles ABC and DEF keeping their relative positions. Show how to map $\triangle ABC$ into $\triangle DEF$ by the method used in part (a) .



8. The points on the following curves are rotated through an angle of measure $\frac{\pi}{6}$ with respect to the origin. Find the equations of the transformed curves. Sketch each of the curves and its image on the same set of axes.

(a) $3x + 2y - 8 = 0$

(b) $x^2 + y^2 = 25$

(c) $y^2 = 4x$

9. Discuss the transformation $(x,y) \longrightarrow (-y + 3 , x + 1)$ by finding the images of the curves in Exercise 8.

10. Determine whether parallelism is preserved when the lines $L_1 : 3x - 2y + 5 = 0$ and $L_2 : 3x - 2y - 3 = 0$ undergo the mapping $(x,y) \longrightarrow (x + y , 2x - y)$.

10-6. Inversions.

We conclude with a discussion of a point transformation called an inversion.

Consider a circle C with radius r and center at O . Select any point $P \neq O$, $d(O,P) \geq \frac{1}{2}r$, and draw $\overleftrightarrow{OP}$. With P as a center and $\overline{OP}$

as radius draw an arc intersecting C at R. Finally, with R as center and a radius r draw an arc intersecting $\overleftrightarrow{OP}$ in P'. The construction is shown in Figure 10-19. (Note that this construction requires that the circle be intersected at point R.)

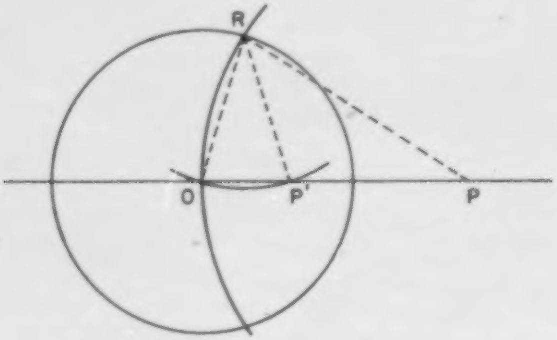


Figure 10-19

$\triangle ORP$ is isosceles since $\overline{OP} \cong \overline{RP}$; $\triangle ORP'$ is isosceles since $\overline{OR} \cong \overline{RP'}$. Thus $\angle ORP \cong \angle POR \cong \angle OP'R$ and $\triangle RPO \cong \triangle ORP'$. Then $\frac{d(O,P)}{d(O,R)} = \frac{d(O,R)}{d(O,P')}$ and $d(O,P) \cdot d(O,P') = r^2$. Two points P and P' which meet this condition are said to be mutually inverse points with respect to circle C.

When $d(O,P) < \frac{1}{2}r$, the arc drawn with P as a center and $\overline{OP}$ as radius will not intersect the circle. In this case, construct the perpendicular bisector of $\overline{OP}$ intersecting the circle at R and $\overline{OP}$ in S. At R, construct $\angle ORT \cong \angle POR$. Then $\overrightarrow{RT}$ will intersect $\overleftrightarrow{OP}$ in P'. It is left as an exercise to prove that $\overline{OP} \cdot \overline{OP'} = r^2$.

> DEFINITION. An inversion is a point transformation which maps each of two arbitrary points which are mutually inverse into the other.

Circle C is called the circle of inversion and point O is called the center of inversion. Point P' is said to be the inverse or image of P, and vice-versa.

Each point on the unit circle is its own image; each point outside this circle has a unique image inside; and, with the exception of the origin, each point inside the circle has a unique image outside. This is true because if $d(O,P) < r$, we have $d(O,P') > r$, and for $d(O,P) > r$, we have $d(O,P') < r$. For any point on the unit circle, $d(O,P) = d(O,P') = r$.

We now obtain an analytic representation for such a transformation. For simplicity, we let $r = 1$.

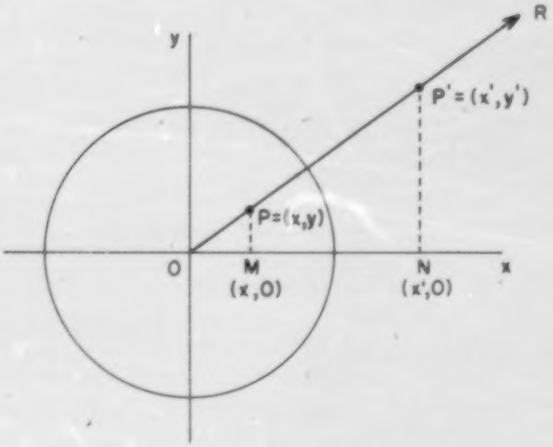


Figure 10-20

Given a unit circle C with its center at the origin. Draw any ray $\overrightarrow{OR}$ and locate on $\overrightarrow{OR}$ mutually inverse points P and P'. Construct perpendiculars from P and P' to the x-axis, intersecting the axis at M and N respectively.

(1) Since $\triangle OMP \cong \triangle ONP'$, $\dfrac{d(O,P)}{d(O,P')} = \dfrac{x}{x'}$.

(2) By definition, we have $d(O,P) \cdot d(O,P') = 1$ or $d(O,P) = \dfrac{1}{d(O,P')}$.

(3) Thus by substitution, $\dfrac{x}{x'} = \dfrac{1}{(d(O,P'))^2} = (d(O,P))^2$.

(4) Since $(d(O,P))^2 = x^2 + y^2$ and $(d(O,P'))^2 = x'^2 + y'^2$,

we have $\dfrac{x}{x'} = \dfrac{1}{x'^2 + y'^2}$ and $\dfrac{x}{x'} = x^2 + y^2$.

(5) Thus $x = \dfrac{x'}{x'^2 + y'^2}$ and $x' = \dfrac{x}{x^2 + y^2}$.

(6) In a similar fashion, $y = \frac{y'}{x'^2 + y'^2}$ and $y' = \frac{y}{x^2 + y^2}$.

(7) The pairs of equations:

$$\begin{cases} x = \dfrac{x'}{x'^2 + y'^2} \\ y = \dfrac{y'}{x'^2 + y'^2} \end{cases} \quad \text{and} \quad \begin{cases} x' = \dfrac{x}{x^2 + y^2} \\ y' = \dfrac{y}{x^2 + y^2} \end{cases}$$

are called the equations of the inversion transformation. We shall now investigate the effect of applying this transformation to several curves.

Example 1. What is the inverse of a straight line with respect to a unit circle?

(1) Let $L : ax + by + c = 0$ with $c \neq 0$. Then L' , the inverse of L , has the equation

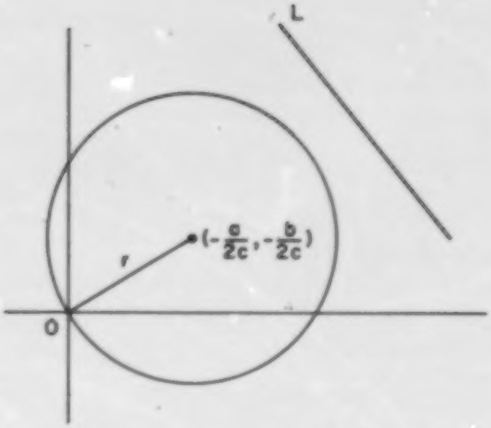
$$\frac{ax'}{x'^2 + y'^2} + \frac{by'}{x'^2 + y'^2} + c = 0 .$$

(2) Thus $c(x'^2 + y'^2) + ax' + by' = 0$, or $x'^2 + y'^2 + \frac{a}{c}x' + \frac{b}{c}y' = 0$.

(3) Completing the squares, we have:

$$\left(x' + \frac{a}{2c}\right)^2 + \left(y' + \frac{b}{2c}\right)^2 = \frac{a^2 + b^2}{4c^2}$$

and we recognize the graph of a circle with center at $\left(-\frac{a}{2c} , -\frac{b}{2c}\right)$, with $r = \frac{\sqrt{a^2 + b^2}}{2c}$, and passing through the origin as illustrated in Figure 10-21 .



$$r = \frac{\sqrt{a^2 + b^2}}{2c}$$

Figure 10-21

Thus a line not passing through the origin transforms into a circle passing through the origin. The converse of this theorem is also true: a circle passing thru the origin transforms into a straight line not passing through the origin. The proof is left as a Challenge Problem.

There is an interesting special case of this problem. Note that in the example given we defined the line L by $ax + by + c = 0$ and $c \neq 0$. What if $c = 0$?

In this case, we have $L : ax + by = 0$ or $y = -\frac{a}{b}x$ or $y = mx$ where m is the slope. The inversion transformation yields

$$\frac{y'}{x'^2 + y'^2} = \frac{mx'}{x'^2 + y'^2} .$$

Thus $y' = mx'$ and we observe that a line passing through the origin transforms into itself. Another way of saying this is that a line passing through the origin remains invariant under an inversion transformation.

Example 2. What is the inverse, with respect to the unit circle, of a network of lines $x = c$, parallel to the y-axis, and $y = k$, parallel to the x-axis?

(1) The lines $x = c$ transform into

$$\frac{x'}{x'^2 + y'^2} = c \quad \text{or} \quad c(x'^2 + y'^2) = x'$$

(2) Thus $x'^2 + y'^2 - \frac{x'}{c} = 0$, or $(x' - \frac{1}{2c})^2 + y'^2 = \frac{1}{4c^2}$.

This equation represents a whole "family of circles" passing through the origin with centers at $(\frac{1}{2c}, 0)$.

(3) In a similar fashion, the lines $y = k$ transform into a family of circles with centers at $(0, \frac{1}{2k})$ and passing through the origin.

A part of a network of lines and the circles which are their inverses are shown in Figure 10-22.

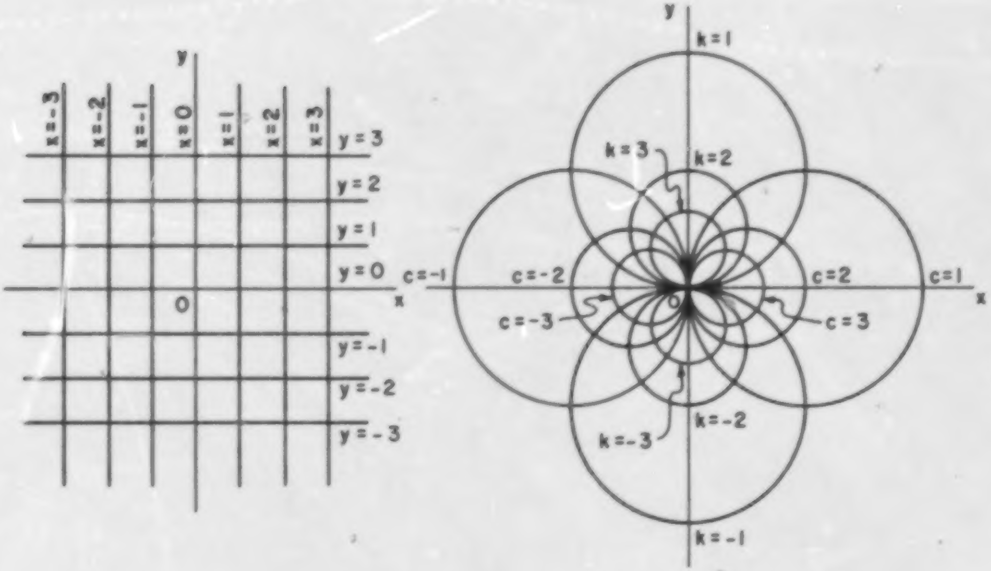

y
x=-3
x=-2
x=-1
x=0
x=1
x=2
x=3
y=3
y=2
y=1
y=0
x
0
y=-1
y=-2
y=-3
y
k=1
k=3
k=2
c=-1
c=-2
c=2
c=1
x
c=-3
0
c=3
k=-3
k=-2
k=-1

Figure 10-22

You have already observed an unusual result. For the first time in this discussion, a curve has been transformed into a different curve. Such an event was made possible because we are dealing with point transformations. In Figure 10-20, a different scale was used for the two drawings.

As a final example, we consider the following;

Example 3. What is the inverse of a circle with respect to the unit circle?

(1) Consider the general equation of a circle

$C : x^2 + y^2 + Dx + Ey + F = 0$, and apply the equations of inversion. Thus we have

$$\frac{x'^2}{(x'^2+y'^2)^2} + \frac{y'^2}{(x'^2+y'^2)^2} + \frac{Dx'}{x'^2+y'^2} + \frac{Ey'}{x'^2 \quad y'^2} + F = 0$$

or $$\frac{1}{x'^2 + y'^2} + \frac{Dx'}{x'^2 + y'^2} + \frac{Ey'}{x'^2 + y'^2} + F = 0$$

(2) Thus since $x'^2 + y'^2 \neq 0$, $F(x'^2 + y'^2) + Dx' + Ey' + 1 = 0$

or $x'^2 + y'^2 + \frac{D}{F}x' + \frac{E}{F}y' + \frac{1}{F} = 0$

(3) Substituting $D' = \frac{D}{F}$, $E' = \frac{E}{F}$, $F' = \frac{1}{F}$, we get

$$C' : x'^2 + y'^2 + D'x' + E'y' + F' = 0$$

which we recognize as a different circle (in general).

It may be of interest to discover whether C and C' are related to each other in any way.

Exercises 10-6

The first five exercises are concerned with the effect of inverting the given curve with respect to the unit circle. The equations of the inversion are

$$x = \frac{x'}{x'^2 + y'^2} , \quad y = \frac{y'}{x'^2 + y'^2} .$$

For each exercise, draw the circle of inversion, the original curve, and its inverse on the same graph.

1. $3x + 2y - 6 = 0$
2. $y = 5x$

3. $y = 3$

4. $y^2 = 4x$ (The graph of the inverted curve is optional)

5. $(x - 4)^2 + (y - 4)^2 = 16$

6. Find the inverse of each of the following lines with respect to the unit circle. Graph all of them on one set of axes and all their inverses on another set. The lines are: $x = \pm 2$, $x = \pm 4$, $x = \pm 6$, $y = \pm 2$, $y = \pm 4$, and $y = \pm 6$.

7. In Exercise 1 you found the inverse of the line $L : 3x + 2y - 6 = 0$. Call the inverse L' . Now apply the same transformation to L' . What can you conjecture from the result?

8. Derive equations of inversion with respect to a circle whose radius is r and center at the origin.

9. The following four points are collinear: $A = (0,-3)$, $B = (1,-1)$, $C = (2,1)$ and $D = (3,3)$. Find the inverse of each of these points with respect to the circle $x^2 + y^2 = 4$ and call the inverse points A' , B' , C' , and D' . Prove that

$$\frac{\dfrac{d(A,C)}{d(A,D)}}{\dfrac{d(B,C)}{d(B,D)}} = \frac{\dfrac{d(A',C')}{d(A',D')}}{\dfrac{d(B',C')}{d(B',D')}}$$

(This ratio is called a cross-ratio in more advanced geometries).

10. Refer to the text and perform the construction of the inverse point P' when $r < \frac{1}{2}$. Prove that $\overline{OP} \cdot \overline{OP'} = r^2$.

Challenge Problem

Prove that a circle passing through the origin inverts into a straight line not passing through the origin.

10-7. Summary and Review Exercises.

We have considered two types of geometric transformations. The first type considered a transformation as an operation which changed one set of axes into another by means of translation or a rotation or both. In a translation, the axes are shifted in such a way that they remain parallel to their original positions and oriented in the same direction; the origin is moved. In contrast, a rotation keeps the origin fixed but new axes are obtained by rotating the axes through a fixed angle. Sets of equations were derived to effect these operations. We demonstrated how a relatively complex equation could be reduced to a simpler form which then could be drawn more readily.

As a second type of transformation we considered the mapping of the plane onto itself. Rules were given by which any point or sets of points in the plane can be moved from one position to another. This set of transformations can effect translations and rotations. It can also effect reflections, inversions, and other changes. Reflections are related to the concept of symmetry in figures. Inversions can convert one type of curve into another. The exercises illustrated some other types of point transformations.

One of the principal reasons for studying transformations is to discover which geometric properties remain invariant under the stated operations. Geometries are classified on the basis of these properties. Euclidean geometry is characterized by the fact that the measures of distance and angle are invariant under the set of all rotations and translations. This set is often referred to as the set of rigid motions since those transformations preserve size and shape. Other invariant properties were considered in the exercises.

Review Exercises

PART II

The "Review Exercises" are concerned primarily with several transformations not discussed in the text. They are presented so that you may discover some significant facts for yourself and may widen your experience with the subject.

1. Find the curve into which the parabola $x^2 = 2y$ is transformed by each of the following mappings:

   (a) $(x,y) \longrightarrow (2x,3y)$

   (b) $(x,y) \longrightarrow (x + 2, 3y)$

   (c) $(x,y) \longrightarrow (x - 1, y + 2)$.

   Draw the original curve and its image for each. Can you find any invariant properties under any of these transformations?

2. The mapping $(x,y) \rightarrow (kx,ky)$ is called the transformation of similitude. Let $k = 2$ and find the effect of this transformation upon the graphs of the following:

(a) $2x + 3y - 6 = 0$

(b) $x^2 + y^2 = 25$

(c) $y^2 = -4x$

Which are invariant properties under this transformation? Can you justify the name given to this transformation?

3. The transformation $T : \begin{cases} x = \dfrac{x' + y'}{2} \\ y = \dfrac{x' - y'}{2} \end{cases}$ is applied to the perpendicular lines $L_1 : 2x - 3y + 4 = 0$ and $L_2 : 3x + 2y - 6 = 0$. Determine whether the geometric property of perpendicularity is preserved under T.

4. The set of affine transformations is one of the most fruitful of all types studied by mathematicians. They have the form

$T \begin{cases} x = ax' + by' + c \\ y = dx' + ey' + f \end{cases}$. Many of the mappings studied in this chapter were special cases of this set. For example, the set of rotations are derived by letting the constants $a = \cos\theta$, $b = -\sin\theta$, $c = 0$, $d = \sin\theta$, $e = \cos\theta$ and $f = 0$.

Consider the special case: $T \begin{cases} x = 2x' - 4y' + 1 \\ y = 3x' + 2y' - 4 \end{cases}$ and find its effect upon the graphs of the following:

(a) $x^2 + y^2 = 4$

(b) $4x^2 - 9y^2 = 36$

(c) $4x - 3y + 12 = 0$

(d) $4x - 3y - 1 = 0$

(You probably cannot identify the images of (a) and (b) unless you study the Supplement to Chapter 7.)

5. In Problem 4, construct lines (c) and (d) and their images on the same set of coordinates. What tentative conclusion can you draw?

6. Prove that the mapping $(x,y) \longrightarrow (-x,-y)$ is a distance preserving transformation.

Table I

Natural Trigonometric Functions (Degree Measure)

Deg.	Sine	Cosine	Tangent	Cotangent	
0	0.000	1.000	0.000	*****	90
1	0.017	1.000	0.017	57.29	89
2	0.035	0.999	0.035	28.64	88
3	0.052	0.999	0.052	19.08	87
4	0.070	0.998	0.070	14.30	86
5	0.087	0.996	0.087	11.43	85
6	0.105	0.995	0.105	9.514	84
7	0.122	0.993	0.123	8.144	83
8	0.139	0.990	0.141	7.115	82
9	0.156	0.988	0.158	6.314	81
10	0.174	0.985	0.176	5.671	80
11	0.191	0.982	0.194	5.145	79
12	0.208	0.978	0.213	4.705	78
13	0.225	0.974	0.231	4.331	77
14	0.242	0.970	0.249	4.011	76
15	0.259	0.966	0.268	3.732	75
16	0.276	0.961	0.287	3.487	74
17	0.292	0.956	0.306	3.271	73
18	0.309	0.951	0.325	3.078	72
19	0.326	0.946	0.344	2.904	71
20	0.342	0.940	0.364	2.747	70
21	0.358	0.934	0.384	2.605	69
22	0.375	0.927	0.404	2.475	68
23	0.391	0.921	0.424	2.356	67
24	0.407	0.914	0.445	2.246	66
25	0.423	0.906	0.466	2.145	65
26	0.438	0.899	0.488	2.050	64
27	0.454	0.891	0.510	1.963	63
28	0.469	0.883	0.532	1.881	62
29	0.485	0.875	0.554	1.804	61
30	0.500	0.866	0.577	1.732	60
31	0.515	0.857	0.601	1.664	59
32	0.530	0.848	0.625	1.600	58
33	0.545	0.839	0.649	1.540	57
34	0.559	0.829	0.675	1.483	56
35	0.574	0.819	0.700	1.428	55
36	0.588	0.809	0.727	1.376	54
37	0.602	0.799	0.754	1.327	53
38	0.616	0.788	0.781	1.280	52
39	0.629	0.777	0.810	1.235	51
40	0.643	0.766	0.839	1.192	50
41	0.656	0.755	0.869	1.150	49
42	0.669	0.743	0.900	1.111	48
43	0.682	0.731	0.933	1.072	47
44	0.695	0.719	0.966	1.036	46
45	0.707	0.707	1.000	1.000	45
	Cosine	Sine	Cotangent	Tangent	Deg.

Table II

Natural Trigonometric Functions (Radian Measure)

Rad.	Sine	Cosine	Tangent	Cotangent
.00	0.000	1.000	0.000	*****
.02	0.020	1.000	0.020	49.99
.04	0.040	0.999	0.040	24.99
.06	0.060	0.998	0.060	16.65
.08	0.080	0.997	0.080	12.47
.10	0.100	0.995	0.100	9.967
.12	0.120	0.993	0.121	8.293
.14	0.140	0.990	0.141	7.096
.16	0.159	0.987	0.161	6.197
.18	0.179	0.984	0.182	5.495
.20	0.199	0.980	0.203	4.933
.22	0.218	0.976	0.224	4.472
.24	0.238	0.971	0.245	4.086
.26	0.257	0.966	0.266	3.759
.28	0.276	0.961	0.288	3.478
.30	0.296	0.955	0.309	3.233
.32	0.315	0.949	0.331	3.018
.34	0.333	0.943	0.354	2.827
.36	0.352	0.936	0.376	2.657
.38	0.371	0.929	0.399	2.504
.40	0.389	0.921	0.423	2.365
.42	0.408	0.913	0.447	2.239
.44	0.426	0.905	0.471	2.124
.46	0.444	0.896	0.495	2.018
.48	0.462	0.887	0.521	1.921
.50	0.479	0.878	0.546	1.830
.52	0.497	0.868	0.573	1.747
.54	0.514	0.858	0.599	1.668
.56	0.531	0.847	0.627	1.595
.58	0.548	0.836	0.655	1.526
.60	0.565	0.825	0.684	1.462
.62	0.581	0.814	0.714	1.401
.64	0.597	0.802	0.745	1.343
.66	0.613	0.790	0.776	1.289
.68	0.629	0.778	0.809	1.237
.70	0.644	0.765	0.842	1.187
.72	0.659	0.752	0.877	1.140
.74	0.674	0.738	0.913	1.095
.76	0.689	0.725	0.950	1.052
.78	0.703	0.711	0.989	1.011
.80	0.717	0.697	1.030	0.971
.82	0.731	0.682	1.072	0.933
.84	0.745	0.667	1.116	0.896
.86	0.758	0.652	1.162	0.861
.88	0.771	0.637	1.210	0.827
.90	0.783	0.622	1.260	0.794

Table II

Natural Trigonometric Functions (Radian Measure)

Rad.	Sine	Cosine	Tangent	Cotangent
.92	0.796	0.606	1.313	0.761
.94	0.808	0.590	1.369	0.730
.96	0.819	0.574	1.428	0.700
.98	0.830	0.557	1.491	0.671
1.00	0.841	0.540	1.557	0.642
1.02	0.852	0.523	1.628	0.614
1.04	0.862	0.506	1.704	0.587
1.06	0.872	0.489	1.784	0.560
1.08	0.882	0.471	1.871	0.534
1.10	0.891	0.454	1.965	0.509
1.12	0.900	0.436	2.066	0.484
1.14	0.909	0.418	2.176	0.460
1.16	0.917	0.399	2.296	0.436
1.18	0.925	0.381	2.427	0.412
1.20	0.932	0.362	2.572	0.389
1.22	0.939	0.344	2.733	0.366
1.24	0.946	0.325	2.912	0.343
1.26	0.952	0.306	3.113	0.321
1.28	0.958	0.287	3.341	0.299
1.30	0.964	0.268	3.602	0.278
1.32	0.969	0.248	3.903	0.256
1.34	0.973	0.229	4.256	0.235
1.36	0.978	0.209	4.673	0.214
1.38	0.982	0.190	5.177	0.193
1.40	0.985	0.170	5.798	0.172
1.42	0.989	0.150	6.581	0.152
1.44	0.991	0.130	7.602	0.132
1.46	0.994	0.111	8.989	0.111
1.48	0.996	0.091	10.98	0.091
1.50	0.997	0.071	14.10	0.071
1.52	0.999	0.051	19.67	0.051
1.54	1.000	0.031	32.46	0.031
1.56	1.000	0.011	92.62	0.011
1.58	1.000	-0.009	-108.65	-0.009
1.60	1.000	-0.029	-34.23	-0.029
1.62	0.999	-0.049	-20.31	-0.049
1.64	0.998	-0.069	-14.43	-0.069
1.66	0.996	-0.089	-11.18	-0.089
1.68	0.994	-0.109	-9.121	-0.110
1.70	0.992	-0.129	-7.697	-0.130
1.72	0.989	-0.149	-6.652	-0.150
1.74	0.986	-0.168	-5.853	-0.171
1.76	0.982	-0.188	-5.222	-0.191
1.78	0.978	-0.208	-4.710	-0.212
1.80	0.974	-0.227	-4.286	-0.233

The Greek Alphabet

Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	xi
Γ	γ	gamma	Ο	ο	omicron
Δ	δ	delta	Π	π	pi
Ε	ϵ	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	ϒ	υ	upsilon
Ι	ι	iota	Φ	φ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

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For precisely defined analytic geometry terms the reference is to the formal definition. For other terms the reference is to an informal definition or to the most prominent discussion.

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Supplement A

DETERMINANTS

If we suppose that this system of equations has a solution:

$$ax + by = c$$
$$px + qy = r\ ,$$

it can be found by elementary methods to be:

$$x = \frac{cq - br}{aq - bp}\ , \quad y = \frac{ar - cp}{aq - bp}\ .$$

These numerators and denominators may be written in a form which helps to develop a useful algebraic concept and notation:

$$x = \frac{\begin{vmatrix} c & b \\ r & q \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}\ , \quad y = \frac{\begin{vmatrix} a & c \\ p & r \end{vmatrix}}{\begin{vmatrix} a & b \\ p & q \end{vmatrix}}$$

An expression of the form $\begin{vmatrix} a & b \\ p & q \end{vmatrix}$ is called a *determinant*, and its value, as suggested by the example above, is defined:

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix} = aq - bp$$

This determinant has two rows: a,b; and p,q; and two columns: a,p; and b,q . It is called a *second order determinant*, and has $4 = 2^2$ terms or elements. A *third order determinant* has three rows and three columns, and $9 = 3^2$ elements. A determinant of order n has n rows and n columns, and so on. We frequently use "Δ" to indicate either a determinant or its value. Note that the first order determinant $|a|$ has the value a .

We list a number of theorems, all of which are true for determinants of any order, and indicate briefly proofs for the second order. In most cases the proof for higher orders is a straightforward generalization of the proof for the second order.

THEOREM 1. Δ is unchanged if we interchange rows with columns.

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix} = \begin{vmatrix} a & p \\ b & q \end{vmatrix} = aq - bp = \Delta$$

Note: All these theorems remain valid if we interchange the words "row", "column."

THEOREM 2. If two rows of Δ are interchanged, the sign of Δ is changed.

$$\begin{vmatrix} p & q \\ a & b \end{vmatrix} = bp - aq = -(aq - bp) = -\Delta .$$

THEOREM 3. If every element of a row of Δ is multiplied by k, then so is Δ.

$$\begin{vmatrix} ka & kb \\ p & q \end{vmatrix} = kaq - kbp = k(aq - bp) = k\Delta .$$

THEOREM 4. If two rows of Δ are equal or proportional, then $\Delta = 0$.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0 , \quad \begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = k\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 .$$

THEOREM 5. Two determinants may be added if they agree in all the elements of $n - 1$ rows. Their sum is then a determinant with these same $n - 1$ rows; and the elements of the remaining row are the sums of the corresponding elements in the original determinants.

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix} + \begin{vmatrix} c & d \\ p & q \end{vmatrix} = aq - bp + cq - dp = (a + c)q - (b + d)p$$

$$= \begin{vmatrix} a + c & b + d \\ p & q \end{vmatrix} .$$

THEOREM 6. A determinant is unchanged if, to the elements of any row we add a common multiple of the corresponding elements of another row.

$$\begin{vmatrix} a + kp & b + kq \\ p & q \end{vmatrix} = \begin{vmatrix} a & b \\ p & q \end{vmatrix} + \begin{vmatrix} kp & kq \\ p & q \end{vmatrix} = \Delta + 0 = \Delta .$$

Notation. It is convenient, for purposes of generalization, to use "double subscript notation."

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

where "a_{ij}" designates the element in row i and column j.

Exercise. Rewirte the proofs of Theorems 1-6 using double subscript notation.

DEFINITION. Minor of a_{ij} (Notation A_{ij}) is the determinant of the square array obtained by removing from Δ all elements of row i, and of column j; we sometimes use the same word to indicate the value of that determinant. Note that A_{ij} is of order $n - 1$.

DEFINITION. Cofactor of a_{ij} (Notation α_{ij}) $\alpha_{ij} = (-1)^{i+j} A_{ij}$. Note that α_{ij} is the same as A_{ij} if the sum of its row and column numbers is even, and α_{ij} is the negative of A_{ij} if the sum of its row and column numbers is odd. As above, we use "cofactor" to indicate the expression as well as its value.

Example 1.

$$\begin{vmatrix} a & b \\ p & q \end{vmatrix}$$

The minor of a is q; of p is b.
The cofactor of a is q; of p is $-b$.

Example 2.

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$$

The minor of p is $\begin{vmatrix} b & c \\ v & w \end{vmatrix}$, of c is $\begin{vmatrix} p & q \\ u & v \end{vmatrix}$.

The cofactor of p is $(-1)^{2+1} \begin{vmatrix} b & c \\ v & w \end{vmatrix} = -\begin{vmatrix} b & c \\ v & w \end{vmatrix}$.

The cofactor of c is $(-1)^{1+3} \begin{vmatrix} p & q \\ u & v \end{vmatrix} = \begin{vmatrix} p & q \\ u & v \end{vmatrix}$.

Example 3.

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{vmatrix}$$

The minor of 8 is $\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$ which has the value $21 - 24 = -3$.

The cofactor of 8 is $(-1)^{3+1}$ times the minor of 8 , and also has the value -3.

The minor of 9 is $\begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix}$ which has the value $14 - 20 = -6$.

The cofactor of 9 is $(-1)^{3+2}$ times the minor of 9 , and has the value 6 .

Exercise. Find the cofactors of each of the nine elements of (3) above, or by applying Theorem 6 to write the determinant in a form simpler to evaluate, thus:

(1) Write the same second column, then add (-2) times these elements to the corresponding element of the third column; then add (-4) times these same elements to the corresponding element of the first column:

$$\begin{vmatrix} 4(-4) + 3 & 4 & 4(-2) + 1 \\ 3(-4) + 2 & 3 & 3(-2) + 5 \\ 1(-4) + 4 & 1 & 1(-2) + 2 \end{vmatrix}$$

which yields the equal determinant

$$\begin{vmatrix} -13 & 4 & -7 \\ -10 & 3 & -1 \\ 0 & 1 & 0 \end{vmatrix} .$$

If we now evaluate by using the element of the third row, we get

$$0\begin{vmatrix} 4 & -7 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} -13 & -7 \\ -10 & -1 \end{vmatrix} + 0\begin{vmatrix} -13 & 4 \\ -10 & 3 \end{vmatrix} = 0 - 1(13 - 70) + 0 = -1(-57) = 57$$

DEFINITION. The value of any determinant is equal to the sum of the products of the elements of the first row by their corresponding cofactors. Application: Cramer's Rule?

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c| = ad - bc .$$

$$\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = a\begin{vmatrix} q & r \\ v & w \end{vmatrix} - b\begin{vmatrix} p & r \\ u & w \end{vmatrix} + c\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

$$= a(qw - rv) - b(pv - ru) + c(pv - qu)\ ,\ \text{etc.}$$

Example.

$$\begin{vmatrix} 3 & 4 & 1 \\ 2 & 3 & 5 \\ 4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 3(6 - 5) - 4(4 - 20) + 1(2 - 12)$$

$$= 3(1) - 4(-16) + 1(-10)$$

$$= 3 + 64 - 10 = 57$$

Notation.

$$\sum_{j=1}^{n} A_{kj}\alpha_{ij}$$

MAIN THEOREM. The value of a determinant is equal to the sum of the products of the elements of any row by their corresponding cofactors.

The proof of this Main Theorem must be carried on by induction and is sufficiently difficult to be put off to another course, but the student is urged to write any third order determinant, and to evaluate it in a number of ways. Note that by a judicious application of the theorems above, the process of evaluating a determinant can be considerably shortened, by obtaining equivalent determinants with some zero elements.

Notation. From the Main Theorem:

$$\Delta = \sum_{i=1}^{n} A_{ij}\alpha_{ij} = \sum_{j=1}^{n} A_{ij}\alpha_{ij}\ .$$

Example. We may evaluate the determinant of the example above by using the element of the second row:

$$-2\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} = -2(7) + 3(2) - 5(-13) = -14 + 6 + 65 = 57$$

or of the third column:

$$1\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 1(-10) - 5(-13) + 2(1) = -10 + 65 + 2 = 57$$

Exercises. [I can supply as many as we think necessary.]

Supplement B

FLOW CHART FOR TWO LINEAR EQUATIONS IN X AND Y .

Suppose we want to study the possible geometric relations between the graphs of two linear equations

$$L_1 : a_1x + b_1y + c_1 = 0$$

$$L_2 : a_2x + b_2y + c_2 = 0$$

Suppose further that we want the study to cover all pairs of ordered triples of real numbers (a_1,b_1,c_1) and (a_2,b_2,c_2) . If we agree to include all such pairs, the study can easily be converted to a computer program and the coefficients themselves can even be generated internally in the computer as a part of a larger program.

If we know that the equations are not degenerate (i.e., either the x or y coefficient is different from zero), each represents a line in the plane, and these lines may be identical, parallel or intersecting. What we want to construct is an ordered set of questions we can ask about the coefficients of L_1 and L_2 which will distinguish for us how the graphs would have looked if we had drawn them. Our questions must be phrased in such a way that each answer will be either "yes" or "no."

Of course many different patterns of questions are possible. In general we want the pattern to branch like a tree with each question so that if an answer is "yes", the succeeding path will be different than it would have been had the answer been "no." At the end of each path will be a message stating the correct geometric configuration for the pair of equations with which we started. This type of pattern is often called a flow chart and is a useful tool in computer programming. If you think a little you will see that the well known game of Twenty Questions uses a kind of oral flow chart to solve the problem "What am I thinking of?"

Let us consider what the first question in our series should be. If at least one of the given equations is degenerate, then we do not really have two lines to study. We want to design our pattern to channel such equations aside. Accordingly the first question might be

$$\boxed{\text{Is } (|a_1| + |b_1| \cdot (|a_2| + |b_2|) = 0 \text{ ?}}$$

If the answer is "yes", then we know that either $|a_1| + |b_1| = 0$ or $|a_2| + |b_2| = 0$. In other words at least one equation is not really linear. We place the message "Degenerate equation" and end this path. If the answer to the question was "no", we are assured of two linear equations. What shall we ask next? A possible second question is

$$\boxed{\text{Is } a_1b_2 - b_1a_2 \neq 0 \text{ ?}}$$

Notice that this time we ask whether a certain expression is different from zero. Of the answer is "yes", then we know the lines L_1 and L_2 intersect in a point. We write a message to this effect and close the path. If the answer to the second question is "no", then the two lines must be either parallel or coincident. We need a third question which will distinguish between these two cases. One such question is

$$\boxed{\text{Is } |a_1c_2 - a_2c_1| + |c_1b_2 - c_2b_2 - c_2b_1| = 0 \text{ ?}}$$

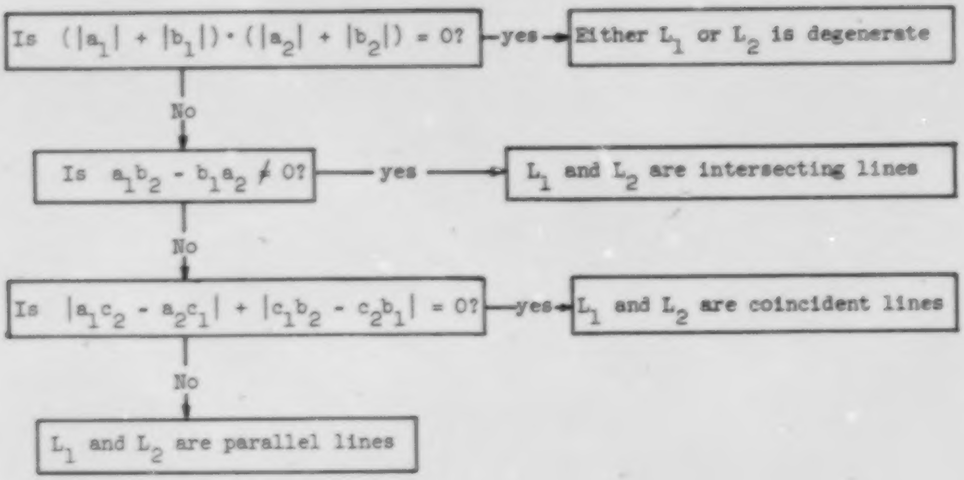
An answer of "yes" guarantees that $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0$ and $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0$. Therefore we have a pair of coincident lines. An answer of "no" in a similar way insures that L_1 and L_2 are parallel.

Let us repeat these three questions together with the message pattern we have indicated.

FLOW CHART

$$L_1 : a_1x + b_1y + c_1 = 0$$

$$L_2 : a_2x + b_2y + c_2 = 0$$

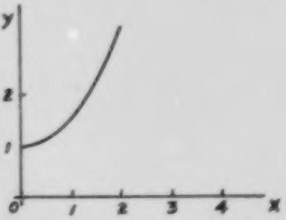

Is $(|a_1| + |b_1|) \cdot (|a_2| + |b_2|) = 0$?
yes
Either L_1 or L_2 is degenerate
No
Is $a_1b_2 - b_1a_2 \neq 0$?
yes
L_1 and L_2 are intersecting lines
No
Is $|a_1c_2 - a_2c_1| + |c_1b_2 - c_2b_1| = 0$?
yes
L_1 and L_2 are coincident lines
No
L_1 and L_2 are parallel lines

Supplement C

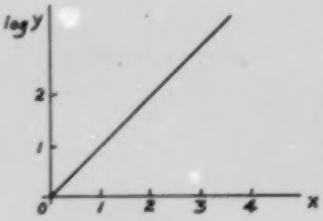
GRAPHS WITH NON-UNIFORM SCALES

In practice, it is sometimes necessary to graph a function $f(x,y)$ in a system of reference in which the axes are perpendicular to each other, but a different unit is used on each axis. For example, if the range of a function is very large compared to the domain, any unit small enough to allow the range to be graphed on a piece of paper will compress the domain too much to be helpful. We can study many properties of such a graph, but we must be careful never to read slopes from it without taking the difference of scale into account.

Other interesting variations of graphing $f(x,y)$ using perpendicular axes are semi-logarithmic and logarithmic graphs which prove to be helpful in applications of mathematics to biology, economics, and other sciences, especially where growth is involved. As an example, let us look at the graph of $y = e^x$ first in regular rectangular and then in semi-logarithmic coordinates.



(a)



(b)

Graph (a) is the familiar exponential function studied in Intermediate Mathematics. If $y = e^x$, then x is the natural logarithm of y or $x = \log y$. Clearly there is a linear relation, not between x and y, but between x and $\log y$. If we treat x as usual, and graph not y but $\log y$ on the vertical axis, we do indeed have a straight line. (See graph b.) This is called a semi-logarithmic graph because one of the axes measures the logarithm of a variable, rather than the variable itself.

If we go one step further and plot the logarithm of x on one axis and the logarithm of y (to the same base) on the other axis, we have a logarithmic graph. This type is used extensively in finding equations to fit experimental data when there is reason to believe the relationship is of the form $y = x^k$. Taking the logarithm of each side we have

$$\log y = k \log x .$$

If we graph our exponential data by measuring $\log y$ on one scale and $\log x$ on the other, we should be able to fit a straight line to the data, and determine k as the slope of the line.

As a matter of fact, if a scientist suspects his data could be described by either $y = a^x$ or $y = x^a$, he can plot the data using semi-logarithmic and full logarithmic coordinates. If either graph appears to be a straight line, his problem is solved. If the semi-logarithmic graph is a straight line, then $\log y = (\log a)x$, the slope is the logarithm of the base a , and the data is related by $y = a^x$. If the double logarithmic scale yields a straight line, then the slope, a , determines the exponent in the equation $y = x^a$ which relates the data.

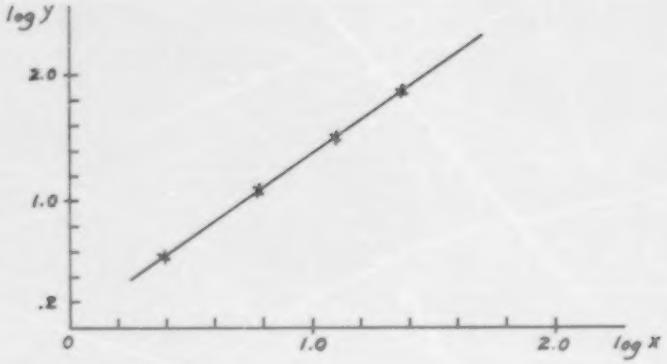
Problem. Suppose you have experimentally determined the following data and want to discover the mathematical relation between x and y .

x	2.50	6.20	11.6	21.4
y	3.61	12.9	30.9	72.9

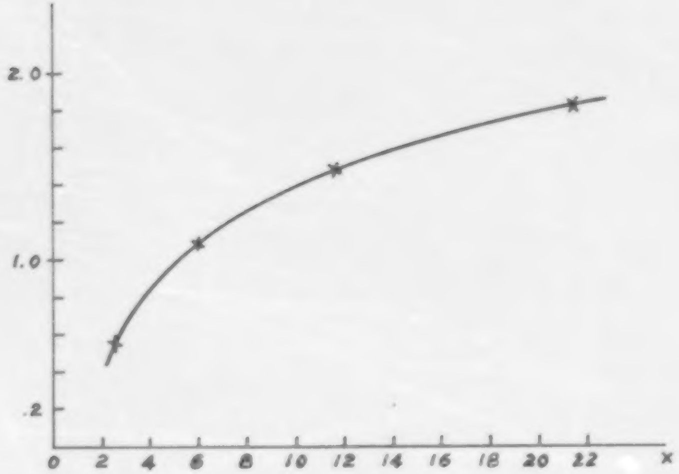
Suppose, further, you guess that y is either an exponential function involving x or that it is a power function of x .

Solution. Using common logarithms we fill out a table and plot the ordered pairs (x,log y) on one graph and (log x, log y) on a second. Then we study the points and if either graph is approximately a straight line, we measure its slope. Finally we use this to express the relation between x and y .

x	2.50	6.20	11.6	21.4
log x	.398	.792	1.06	1.33
y	3.61	12.9	30.9	72.9
log y	.557	1.09	1.49	1.86



GRAPH II



The first graph seems to be linear and its slope $\frac{\Delta y}{\Delta x}$ is approximately $\frac{1.31}{.94} \sim 1.4$. Therefore, $\log y = 1.4 \log x$ or $y = x^{1.4}$ is the relation we are seeking.

Supplement to Chapter 2

COORDINATES AND THE LINE

S2-1.

From the postulates of geometry we deduced immediately that any point on a line may be chosen as the origin for a coordinate system and that the positive coordinates may be assigned to the interior points of either ray determined by the origin. However, in our development of the SMSG *Geometry* there need be no mention of units, in terms of which these measurements are made; the entire development depends upon one intrinsic scale of measure. For this reason we shall describe such coordinate systems as *intrinsic coordinate systems*. It would be very convenient to be free to choose coordinate systems with different scales of measure. It is easy to show that we have this freedom.

The coordinate system is an unusual type of function whose domain is the set of points on the line and whose range is the set of real numbers. Let us denote this function by f, whose value at each point X is the number $f(X) = x$. Let us consider a linear function, g, on the real numbers, defined by the equation $x' = g(x) = ax + b$, where a is any non zero real number and b is any real number. The composite function which assigns to each point X the number $g\big(f(X)\big)$ is also a one-to-one correspondence between the points of the line and the real numbers. We shall describe such correspondences as *linear coordinate systems*. We shall continue to describe the number which corresponds to a point as the *coordinate* of the point, since this phrase has meaning only with reference to a particular coordinate system. We shall denote the composite function of f by g as $g(f)$.

We shall consider the description of the geometric properties of the line in terms of such a linear coordinate system. Is there anything in a linear coordinate system comparable to the measure of distance between two points, R and S, whose coordinates in an intrinsic coordinate system on the line $\overleftrightarrow{RS}$ are r and s respectively? The new coordinates r' and s', of R and S respectively, are related by the equations

$$r' = ar + b$$
$$s' = as + b \ .$$

We discover that

$$\begin{aligned} |r' - s'| &= |(ar + b) - (as + b)| \\ &= |ar - as| \\ &= |a| \cdot |r - s| \ . \end{aligned}$$

Unless $|a| = 1$, $|r' - s'|$ is not equal to $|r - s|$, the measure of distance in the intrinsic coordinate system. However, we do note that in the linear coordinate system, related to the intrinsic coordinate system by the equation $x' = ax + b$, the number $|r' - s'|$ is a constant multiple of $|r - s|$, the constant being independent of the choice of points.

We recall that the length of a segment was defined to be the measure of distance between its endpoints and that congruent segments were defined as segments having the same length. Thus the statement $\overline{RS} \cong \overline{TU}$ is equivalent to the statement, $|r - s| = |t - u|$, where r, s, t, and u are intrinsic coordinates of R, S, T, and U respectively.

If
$$|r - s| = |t - u| \ ,$$
then
$$|a| \cdot |r - s| = |a| \cdot |t - u| \ ,$$
$$|ar - as| = |at - au| \ ,$$
and
$$|(ar + b) - (as + b)| = |(at + b) - (au + b)| \ ,$$
or
$|r' - s'| = |t' - u'|$, where r', s', t', and u' are coordinates in any linear coordinate system. Thus the condition defining congruence for segments applies in any linear coordinate system.

The student should think through all the details of the argument that any linear coordinate system is a one-to-one correspondence between the points of the line and the real numbers. Let f be an intrinsic coordinate system on a line L and let X by any point of L. Then $f(X)$ is a unique real number and so is $g(f(X)) = af(X) + b$. So far we have not used the assumption that $a \neq 0$. Now let r be a real number. Since $a \neq 0$, there is a unique number x_0 such that $ax_0 + b = r$. Since the original coordinate system is a one-to-one correspondence between the points of L and the real numbers, there is a unique point X_0 such that $f(X_0) = x_0$. Hence there is a unique point X_0 on L such that $g(f(X_0)) = g(x_0) = ax_0 + b = r$.

Example. Let P, Q, R, and S be four points on a line with intrinsic coordinates 2, 5, 8, and 11 respectively. Since $|2 - 5| = |8 - 11|$, $\overline{PQ} \cong \overline{RS}$. Let a linear coordinate system be defined by the equation $x' = 2x - 1$. Then the new coordinates of P, Q, R, and S are 3, 9, 15, and 21 respectively. Since $|3 - 9| = |15 - 21|$, the congruence of $\overline{PQ}$ and $\overline{RS}$ is similarly described in terms of the new coordinates.

The other geometric property described in terms of intrinsic coordinate systems on a line is betweenness on the line. We recall that the point S is between R and T if and only if $r < s < t$ or $r > s > t$, where r, s, and t are the coordinates of R, S, and T respectively. We observe that

if $\qquad r < s < t$

then $\qquad ar < as < at$ if $a > 0$, or $ar > as > at$ if $a < 0$

and $\qquad ar + b < as + b < at + b$ if $a > 0$,

or $\qquad ar + b > as + b > at + b$ if $a < 0$.

The members of these inequalities are precisely the coordinates r', s', and t', which would be assigned to the points R, S, and T by a linear coordinate system defined by a linear equation $x' = ax + b$. Thus the last two lines of the above development may be replaced by

$$r' < s' < t' \text{ if } a > 0 \text{, or } r' > s' > t' \text{ if } a < 0 .$$

A similar argument obtains if $r > s > t$. In all cases the condition describing betweenness on a line holds if r, s, and t are replaced by the corresponding coordinates in any linear coordinate system.

The geometric properties of congruence for segments and betweenness on a line are described in exactly the same way in terms of linear coordinate systems as in the intrinsic coordinate systems. We summarize these results from the preceding two paragraphs as follows.

Any intrinsic coordinate system will not be changed under composition with the trivial linear function defined by the equation $x' = x$, and consequently is included among the linear coordinate systems on the line. These are the coordinate systems which are of use and interest to us. Henceforth, we shall usually consider only linear coordinate systems; where there is no chance of ambiguity we shall call these systems coordinate systems.

THEOREM S2-1. If a coordinate system on a line assigns the coordinates r, s, and t to the points R, S, and T, then S is between R and T if and only if $r < s < t$ or $r > s > t$.

THEOREM S2-2. Let P and Q be any two distinct points on a line. In a coordinate system C on the line, the coordinates of P and Q are p and q respectively. Let r and s be any two distinct real numbers. Then there exists a coordinate system C' on the line in which the coordinates of P and Q are r and s respectively.

Proof. We wish to discover whether there exists a linear function which relates C' to C by composition. If there is such a function, there exists an equation $x' = ax + b$ defining the function. The following equations would have to be satisfied.

$$(1) \qquad r = ap + b \quad \text{and} \quad s = aq + b .$$

Combining equations, we obtain

$$r - s = a(p - q)$$

or

$$a = \frac{r - s}{p - q} .$$

Substituting in Equation (1), we obtain

$$r = \frac{r - s}{p - q} \cdot p + b$$

or

$$\begin{aligned} b &= r - \frac{pr - ps}{p - q} \\ &= \frac{pr - qr - pr + ps}{p - q} \\ &= \frac{ps - qr}{p - q} . \end{aligned}$$

The solution set for a and b of this pair of equations is $\{(\frac{r - s}{p - q}, \frac{ps - qr}{p - q})\}$. The coordinate system C' formed by the composition of C by the linear function defined by

$$x' = (\frac{r - s}{p - q})x + \frac{ps - qr}{p - q}$$

does satisfy the conclusion of the theorem. Since $p \neq q$, this equation, and consequently the coordinate system C', is always defined. In C' the coordinates of P and Q are given respectively by

$$p' = \left(\frac{r - s}{p - q}\right)p + \frac{ps - qr}{p - q} = \frac{pr - qr}{p - q} = r$$

and

$$q' = \left(\frac{r - s}{p - q}\right)q + \frac{ps - qr}{p - q} = \frac{ps - qs}{p - q} = s .$$

In fact, the coordinate system C' is unique, though we have not proved it here.

Corollary S2-2-1. If P and Q are any two distinct points on a line with coordinates p and q respectively in a coordinate system C, then the coordinate system C' which is related to C by the linear equation,

$$x' = \frac{1}{q - p} \cdot x - \frac{p}{q - p} ,$$

assigns the coordinates 0 and 1 to the points P and Q respectively. It is sometimes convenient in later computations to write this result in the form $x' = \frac{x - p}{q - p}$.

In order to make intuitively more clear the role played by the constants a and b in the introduction of a new coordinate system, we consider what new coordinates are assigned to the origin and to the unit-point under composition by the linear function defined by the equation $x' = ax + b$.

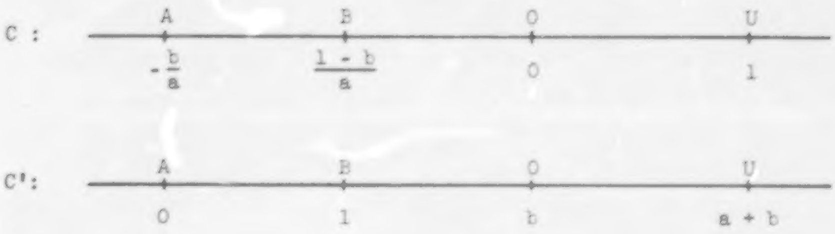


Figure S2-1

The point which was the origin now has coordinate b , and $d(O,U)$, which was 1 is now $|a|$. Thus the role of b is to shift the origin, and one role of a may be to increase or decrease the scale of distance. If $|a| > 1$, we say the new system is *scale-decreasing*; if $|a| < 1$, the new system is *scale-increasing*; if $|a| = 1$, the new system is *scale-preserving*. We observe that if $a > 0$ and the original coordinates p and q of two distinct points are unequal in the order $p < q$, then the new coordinates p' and q' are unequal in the order $p' < q'$, while if $a < 0$ and $p < q$, then p' and q' are unequal in the order $q' < p'$. For these reasons we say that the new system is *order-preserving* if $a > 0$ and *order-reversing* if $a < 0$.

Exercises S2-1a

Let P , Q , and R be points on a line with coordinates -5 , 3 , and 7 respectively. In Problems 1 - 6 find the coordinates of these points in the system given by composition of the original system by the linear function defined by the given equation. Is the new system scale-increasing, scale-decreasing, or scale-preserving? Is it order-preserving or order-reversing?

1. $x' = -x + 3$
2. $x' = 4x - 2$
3. $x' = \frac{1}{4}x + \frac{1}{4}$
4. $x' = -3x$
5. $x' = -\frac{2}{3}x + \frac{7}{3}$
6. $x' = x + 7$
7. For the systems described in Problems 1 - 6 , find the coordinates of the points which were the origins and unit-points in the original system.
8. Find the original coordinates of the points which become the origin and unit-point of the systems described in Problems 1 - 6 .
9. The equation $x' = ax + b$ defining the linear function which relates coordinate systems was subject to the condition $a \neq 0$. Why?

We have not considered the case in which we employ a non-linear equation to define a new coordinate system on a line, but it is interesting to do so. In Problems 10 - 13 the rules defining several functions of other types are given. Examine the coordinate system obtained by the composition of an intrinsic coordinate system and the function defined by the given equation. Does the coordinate system still describe betweenness on the line? Does it describe the congruent segments of the line adequately?

10. $x' = ax^3 + b$

11. $x' = e^x$

12. $\begin{cases} x' = \frac{1}{x} & \text{where } x \neq 0 \\ x' = x & \text{where } x = 0 \end{cases}$

13. $x' = \log_{10} x$

An important mathematical structure which you may have encountered only briefly is the group. A group is a set of elements with a binary operation which has the following properties:

Let S denote the set, a, b, and c, any elements of S, and $\circ$ the binary operation.

(1)	(Closure)	$a \circ b$ is a unique element of S
(2)	(Associativity)	$(a \circ b) \circ c = a \circ (b \circ c)$
(3)	(Identity)	S contains an element e such that $a \circ e = e \circ a = a$
(4)	(Inverse)	For each a there exists a' such that $a \circ a' = a' \circ a = e$.

An element e described in (3) is called an identity and an element a' described in (4) is called an inverse of a .

Some familiar examples of groups are the integers, the rational numbers, or the real numbers with addition as the operation. Other examples are the non-zero rational numbers or non-zero real numbers with multiplication as the operation.

Let us consider the set whose elements are the functions whose domains are the set of real numbers and which are defined by the equations $f(x) = ax + b$ where a is any non-zero real number and b is any real number. This set of functions forms a group under the binary operation of composition.

We shall prove that the identity and inverse properties are satisfied, but we leave the discussion of the closure and associative properties as exercises.

If the set contains an identity, it must be a function defined by a linear equation $g(x) = sx + t$. If this function is an identity, it must satisfy the following equation:

$$f\big(g(x)\big) = g\big(f(x)\big) = f(x) .$$

This becomes $\quad a(sx + t) + b = s(ax + b) + t = ax + b$

or $\quad asx + at + b = sax + sb + t = ax + b$.

This will be true if

(1) $\quad asx = sax = ax$, and

(2) $\quad at + b = sb + t = b$.

Since $a \neq 0$, Equation (1) will be true only if $s = 1$. Equation (2) thus becomes

$$at + b = b + t = b .$$

This equality implies that $t = 0$. Thus, the desired function $g(x) = sx + t = x$. There is only one function of this form. It is in the set, and it can be seen that it is an identity.

Now we want to find inverses. If an element, $f(x) = ax + b$, of the set has an inverse, it must be a function defined by a linear equation $g(x) = sx + t$. If this function is the inverse of $f(x)$, it must satisfy

$$f\big(g(x)\big) = g\big(f(x)\big) = x .$$

This becomes $\quad a(sx + t) + b = s(ax + b) + t = x$

or $\quad asx + at + b = sax + sb + t = x$.

This will be true if

(3) $\quad asx = sax = x$, and

(4) $\quad at + b = sb + t = 0$.

Since $a \neq 0$, Equation (3) will be true if $s = \frac{1}{a}$. Equation (4) becomes

$$at + b = \left(\frac{1}{a}\right) \cdot b + t = 0 ,$$

which is true if $t = \frac{-b}{a}$, which is defined since $a \neq 0$.

Thus the desired function $g(x) = sx + t = (\frac{1}{a})x - \frac{b}{a}$. There is only one function of this form. It is in the set, and it can readily be shown to be an inverse of $f(x)$. In fact, identities and inverses are always unique, but we leave these questions as exercises.

Exercises S2-1b

1. Show that the set and binary operation described above have the closure property.
2. Show that the set and binary operation described above have the associative property.
3. Show that the set and binary operation described above do not have the commutative property.
4. Show that in any group the identity is unique.
5. Show that in any group the inverse of any given element is unique.
6. Show that in any group the inverse of the identity is the identity.
7. Let $f(x) = ax + b$ and $g(x) = px + q$. We denote the inverse of $f(x)$ by $f^{-1}(x)$. Find

   (a) $f\big(f(x)\big)$

   (b) $f\big(g(x)\big)$

   (c) $g\big(f(x)\big)$

   (d) $g\big(g(x)\big)$

   (e) $f\Big(f\big(f(x)\big)\Big)$

   (f) $g\Big(g\big(g(x)\big)\Big)$

   (g) $g^{-1}(x)$

   (h) $f^{-1}\big(g^{-1}(x)\big)$

   (i) $g^{-1}\big(f^{-1}(x)\big)$

   (j) the inverse of $f\big(g(x)\big)$

   (k) $g\big(f^{-1}(x)\big)$

   (ℓ) $f\big(g^{-1}(x)\big)$

*8. Find the function (or functions) $h(x)$ such that

$$h\big(h(x)\big) = f(x) = ax + b \ .$$

Discuss the possibility and number of solutions for $h(x)$.

82-2. Mappings and Linear Transformations.

A function whose domain is a set A and whose range is a set B (which may be the same as A) is frequently called a mapping. An element of the range which corresponds to a given element of the domain is said to be the image of that element. An element of the domain which corresponds to, or is mapped onto, a given element of the range is called a pre-image of that element.

In describing a mapping the second set mentioned may not always be the range of the function, but it always contains the range. If it is the range, the mapping is said to be onto the second set. If the range of the function is a proper subset of the second set, the mapping is said to be into the second set. A mapping is also called a transformation, especially when it is a mapping from a set of geometric entities into a set of geometric entities. The set of images corresponding to the elements of a given set in the domain is called the image set; the set of pre-images corresponding to the elements of a given set in the range is called the pre-image set.

The mappings which we consider in this section are one-to-one transformations of a line onto itself. We consider this line to have a fixed coordinate system. We need such a coordinate system to describe the transformation. We shall consider four types of transformations; translations, reflections, expansions, and contractions.

Intuitively, we may think of a translation as a shifting of the line along itself. A reflection is a half-rotation of the line about the origin. Expansions and contractions are uniform stretching from and shrinking toward the origin. We may describe these more explicitly.

> DEFINITIONS. Let ℓ be a line with a coordinate system; let P be a point on the line with coordinate p ; let the point P' with coordinate p' be the image of P under a transformation of the line ℓ onto itself.
>
> A transformation $T(P) = P'$ is a translation if and only if there exists a real number b such that for every point P , $p' = p + b$.
>
> A transformation $R(P) = P'$ is a reflection if and only if for every point P , $p' = -p$.

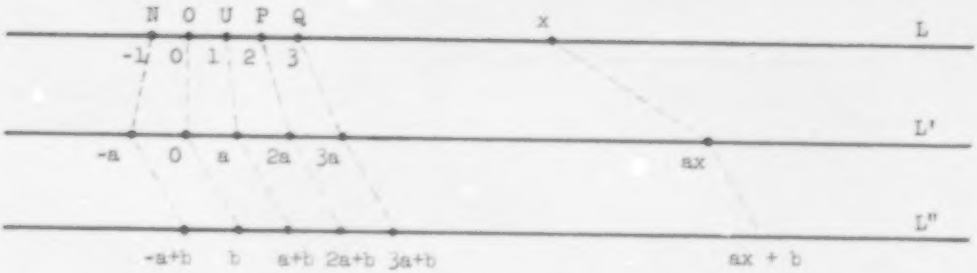
A transformation $E(P) = P'$ is an expansion if and only if there exists a real number $a > 1$ such that for every point P, $p' = ap$.

A transformation $C(P) = P'$ is a contraction if and only if there exists a positive real number $a < 1$ such that for every point P, $p' = ap$.

It should be intuitively apparent that in any of the above transformations an image is between two other images if and only if its pre-image is between the pre-images of the other two images. Therefore, the image set of a segment is also a segment. It should also be apparent that in a translation or a reflection, image segments are congruent if and only if the pre-image segments are congruent. It may or may not be clear that this is also the case in an expansion or contraction. We consider two congruent segments $\overline{PQ}$ and $\overline{RS}$. Their congruence depends upon the equality of $|p - q|$ and $|r - s|$. The congruence of the image segments depends upon the equality of $|p' - q'|$ and $|r' - s'|$. These may be expressed as $|ap - aq| = a|p - q|$ and $|ar - as| = a|r - s|$. These latter numbers are certainly equal if the original segments were congruent. Thus, the image segments of congruent segments are also congruent.

We continue our development by considering compositions of these transformations. A reflection maps a point X onto a point whose coordinate is $-x$; a translation will now map the new point onto a point whose coordinate is $-x + b$. An expansion maps a point X onto a point with coordinate ax; a translation now maps this new point onto a point whose coordinate is $ax + b$.

Such a sequence of transformations may be indicated in a diagram:



It should be understood that L , L' , and L" are the same line, drawn in separate positions to show the transformations clearly. L' is the result of an expansion transformation of L , with the equation $x' = ax$, $(a > 1)$; L" is the result of a translation transformation of L' , with the equation $x'' = x' + b$; finally, L" can be considered as the result of a composition of two transformations of L , with the equation $x'' = ax + b$.

We consider the successive application or composition of two of these transformations and display the results by means of the table below. We employ the notation used in the definitions given above. The labels at the top indicate which transformation is performed first; the labels on the left indicate which transformation is performed second. The entry is the coordinate of the image of a point X , subject to the restrictions of the given transformations. The subscripts of the constants indicate which transformation introduced them.

	T	R	E $(a_1 > 1)$	C $(0 < a_1 < 1)$
T	$x + b_1 + b_2$	$-x + b_2$	$a_1x + b_2$	$a_1x + b_2$
R	$-x - b_1$	x	$-a_1x$	$-a_1x$
E $(a_2 > 1)$	$a_2x + a_2b_1$	$-a_2x$	a_1a_2x	a_1a_2x
C $(0 < a_2 < 1)$	$a_2x + a_2b_1$	$-a_2x$	a_1a_2x	a_1a_2x

We summarize by observing that these transformations and the transformations that may be obtained from them by composition may be included in the set of transformations defined as follows:

> DEFINITION. Let ℓ be a line with a coordinate system; let P be a point on the line with coordinate p ; let the point P' with coordinate p' be the image of P under a transformation of the line ℓ onto itself.
>
> A transformation $T(P) = P'$ is a linear transformation if and only if there exist a non-zero real number a and a real number b such that for every point P , $p' = ap + b$.

We call these mappings linear transformations because the defining equations are linear.

If this argument has not begun to sound familiar, you should go back to Section 2-1.

The set of linear transformations of a line onto itself under the binary operation of composition is another instance of a group.

Exercises S2-2a

In the following exercises, you may find that the form of the proofs you are asked to give are remarkably similar, if not identical, to those in Section 2-1. They are different only in interpretation and terminology.

1. Prove that if Q is between P and R, then in a linear transformation of PR onto itself, the image of Q is between the images of P and R.
2. Prove that if $\overline{PQ}$ and $\overline{RS}$ are congruent segments contained in a line, then in a linear transformation of the line onto itself $\overline{P'Q'} \cong \overline{R'S'}$, where P', Q', R', and S' are the images of P, Q, R, and S respectively.
3. Prove that the set of linear transformations of a line onto itself is closed under composition.
4. Prove that the operation of composition is associative for linear transformations of a line onto itself.
5. Prove that the set of linear transformations of a line onto itself contains an identity with respect to composition.
6. Prove that each element of the set of linear transformations of a line onto itself has an inverse with respect to composition.
7. Prove that the composition of linear transformations of a line onto itself is not commutative. The composition is commutative if certain restrictions are placed on the linear transformations. What are these restrictions?
8. Prove that any linear transformation may be expressed as the composite of not more than three transformations each of which is a translation, a reflection, a contraction, or an expansion.

Although there is no unique way of "factoring" a linear transformation in the way suggested above, it may be that for a given transformation every such expression must include a translation, a reflection, an expansion, or a contraction. In this case we shall say that the linear transformation includes a translation, reflection, or expansion.

We have discovered that the linear transformations of a line onto itself under the binary operation of composition form a group which seems similar to the group of linear functions which describe changes of coordinate system on a line under the binary operation of composition.

This kind of similarity is of some importance in mathematics and is called an isomorphism (from the Greek, ισος, meaning same, and μορφη, meaning form). An isomorphism is a one-to-one correspondence between two mathematical structures which relates not only the elements of the structures but also the operations between the elements. A familiar example is found in the relationship between the multiplication of positive real numbers and the addition of their logarithms. Another example is found in the relationship between the addition of vectors and the addition of complex numbers. The importance of isomorphisms stems from the fact that statements made about one structure may suggest corresponding statements about the other.

In this case the isomorphism is between the group of linear transformations of the line onto itself under composition and the group of changes of coordinate system on the line under composition. The correspondence is established by identical linear functions which occur in the definition of each group. Since our descriptions of each group are in terms of linear functions defined by equations of the form $x' = ax + b$, we may make comparisons of our descriptions when the conditions on a and b are the same.

A change of coordinate system which shifts the origin corresponds to a linear transformation which includes a translation. A change of coordinate system which is measure-preserving corresponds to a linear transformation which includes only a translation or a reflection. A change of coordinate system which is measure-increasing corresponds to a linear transformation which includes a contraction, and a change of coordinate system which is measure-decreasing corresponds to a linear transformation which includes an expansion. A change of coordinate system which is order-preserving corresponds to a linear transformation which does not include a reflection, and a change of coordinate system which is order-reversing corresponds to a linear transformation which includes a reflection.

Lastly, we consider whether a point may be assigned the same coordinate after a change of coordinate system. The comparable situation for a transformation is that a point is mapped onto itself. In either case, where $x' = ax + b$, the situation occurs if $x' = x$.

If $$x' = x\ ,$$

then $$x' = ax + b$$

becomes $$x = ax + b$$

or $$(a - 1)x = -b\ .$$

If $a = 1$ and $b = 0$, we have the identical coordinate system (or the identity transformation) in which all coordinates (or points) are unchanged; if $a = 1$ and $b \neq 0$, there is no coordinate (or point) which is unchanged. If $a \neq 1$, the coordinate (or point with coordinate) $\frac{-b}{a - 1}$ is unchanged. It is customary to say that such numbers or points are fixed or invariant.

Exercises S2-2b

1. Prove that a change of coordinate system is order-preserving if and only if $\frac{r' - s'}{r - s}$ is positive, where r' and s' are the new coordinates of points whose original coordinates were r and s respectively; prove that a change of coordinate system is order-reversing if and only if $\frac{r' - s'}{r - s}$ is negative.

2. Consider a linear transformation of a line onto itself which maps the points R and S, whose coordinates are r and s respectively, onto the points whose coordinates are r' and s' respectively. Prove that the transformation includes:

   (a) a contraction if and only if $0 < \frac{r' - s'}{r - s} < 1$

   (b) a contraction and a reflection if and only if $-1 < \frac{r' - s'}{r - s} < 0$

(c) An expansion if and only if $\frac{r' - s'}{r - s} > 1$

(d) an expansion and a reflection if and only if $\frac{r' - s'}{r - s} < -1$.

3. Consider a linear transformation of a line onto itself which maps the points P and Q , whose coordinates are p and q respectively, onto the points whose coordinates are p' and q' respectively. Prove that the transformation includes:

   (a) a translation if and only if $\frac{p' - q'}{p - q} = 1$

   (b) a reflection if and only if $\frac{p' - q'}{p - q} = -1$.

4. Show that the intrinsic coordinate systems on a line are identical to the linear coordinate systems whose defining functions have the form $x' = x + b$ and $x' = -x + b$, where b is any real number.

5. Consider a line with a coordinate system, let P be a point of the line and let $I(P) = P'$ be the image of P under a transformation of the line onto itself; let p and p' be the coordinates of P and P' respectively.

   Consider the transformation defined by

   $I(P) = P'$ where $p' = \frac{1}{p}$ for $p \neq 0$, and $p' = p$ for $p = 0$.

   Choose an appropriate scale and make a graph for the coordinate system; write the coordinates of several images below. Write the coordinates of their corresponding pre-images above them. A transformation of this type is called an inversion of the line.

6. Consider the composition $F\big(G(H)\big)$ of transformations of a line to itself, where W , X , Y , Z are points of the line with coordinates w , x , y , and z respectively, and

   $F(Y) = Z$ where $z = \frac{1}{y}$ for $y \neq 0$, and $z = y$ for $y = 0$,

   $G(X) = Y$ where $y = x + 1$, and

   $H(W) = X$ where $x = 2^w$.

   (a) Describe the set of pre-images, or domain, and the set of images, or range, of the composite transformation in terms of the coordinate system on the line. Is this transformation into or onto the line? Is this a one-to-one mapping?

(b) Choose an appropriate scale for the coordinate system and make a graph of the set of images of this composite transformation. Write the coordinates of several images below them. Write the coordinates of their corresponding pre-images above them.

(c) Two sets are said to *have the same cardinal number* or *the same cardinality* if their elements may be put in one-to-one correspondence. What can you say about the cardinality of the interior of a segment of a line?

7. Consider the composition $D\ E(F))$ of the functions whose domains are the set of real numbers, where

$$z = D(y) = \begin{cases} \frac{1}{y} & \text{for } y \neq 0 \\ y & \text{for } y = 0 \end{cases}$$

$$y = E(x) = x + 1 \quad \text{for all } x$$

$$x = F(w) = 2^w \quad \text{for all } w .$$

(a) Describe the domain and range of the composite function. Is this mapping into or onto the set of real numbers? Is this mapping one-to-one?

(b) The cardinality of a set is said to be *infinite* if and only if the elements of the set may be put into one-to-one correspondence with the elements of a proper subset of the given set. What can you say about the cardinality of the set of real numbers?

8. If P , Q , R , and S are points, with $R \neq S$, whose respective coordinates in two different coordinate systems are p , q , r , s and p' , q' , r' , s' , prove that

$$\frac{p' - q'}{r' - s'} = \frac{p - q}{r - s} .$$

Each member of the equation is called a *difference quotient*, and in this case expresses the ratio of a pair of directed distances. The content of this theorem might be expressed in this way:

> Difference quotients of directed distances are invariant under a change of coordinate system.

Or this way:

> The ratio of directed distances depends upon the points involved, but not upon the coordinate system.

9. If A, B, and C have respective coordinates 3, 5, and 10 in one coordinate system, and 2, 3, and x in another coordinate system, find x. (In how many ways can you do this problem?)

10. If A, B, and X are distinct points with respective coordinates a, b, x, and a', b', x' in two different coordinate systems, express x' in terms of a, b, a', b', and x.

11. Show that if two points are fixed under a linear transformation, it must be the identity transformation.

Supplement to Chapter 3

LINEAR DEPENDENCE AND INDEPENDENCE

We have defined a zero vector, $\vec{0}$, and, for any number k and vector $\vec{X}$, the scalar product $k\vec{X}$. We may, in the same way, define a zero linear polynomial in one variable, $0 + 0x$; and, for any number k and linear polynomial in one variable, $a + bx$, the "scalar product" $k(a + bx) = ka + kbx$. We could, in the same way, define a zero n-tuple of numbers, and, for any number k and any n-tuple of numbers, the "scalar product", $k(a,b,\ldots,n) = (ka,kb,\ldots,kn)$.

We consider now a set $S = \{A,B,\ldots,K\}$, whose members may all be vectors, or linear polynomials in one variable, or ordered n-tuples of numbers, etc.... We may see that, with suitable definitions along the lines suggested above, members of S might all be linear expressions in two variables, or polynomials in x of degree not greater than 3 , or any polynomials in x , and so on.

A set of such expressions $S = \{A,B,\ldots,K\}$ is said to be linearly dependent (L.D.) if there exists a set of numbers $N = \{a,b,\ldots,k\}$, not all zero, such that $aA + bB + \ldots + kK = 0$.

Example. The set $\{2p + 3q , 6p + 9q\}$ is L.D. because there is a set of numbers $\{-3,1\}$ not all zero, such that $-3(2p + 3q) + 1(6p + 9q) = 0$.

If a set of expressions is not linearly dependent, it is said to be linearly independent (L.I.) .

Example. The set $\{2p + 3q , 6p + 10q\}$ is L.I. because, if there were a set of numbers (a,b) such that $a(2p + 3q) + b(6p + 10q) = 0$, then we would have

$$(2a + 6b)p + (3a + 10b)q = 0$$

for all p and q , or

$$2a + 6b = 0 \quad \text{and} \quad 3a + 10b = 0 .$$

The only solutions for these equations are $a = 0$, $b = 0$; therefore, the original set is not L.D., it is L.I.

In view of the example above, it is possible to define *linear independence* first, as some authors do.

A set of such expressions as $S = \{A,B,\ldots,K\}$ is said to be *linearly independent* (L.I.) if, for the set of numbers $N = \{a,b,\ldots,k\}$, the statement $aA + bB + \ldots + kK = 0$, implies $a = b = \ldots = k = 0$.

Terminology. The property of being L.D. or L.I. is a collective one, and attaches to the *set*, rather than to the separate individuals; however, we follow general usage in writing, sometimes, "The vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are L.I." for the longer "The set of vectors $\{\vec{A},\vec{B},\vec{C}\}$ is L.I."

We state some useful theorems whose proofs are left to the reader.

THEOREM 1. A set is L.D. if any subset of it is L.D.

THEOREM 2. If a set with at least two members is L.D., then one member can be expressed as a linear combination of the others.

Corollary. If the set $\{A,B,\ldots,K\}$ is L.I., and the set $\{A,B,\ldots,K,L\}$ is L.D., then L can be expressed as a linear combination of A , B , ... , K .

THEOREM 3. If the set of rows (or columns) of a determinant is L.D., then the value of the determinant is zero.

Proof. If the set of rows is L.D., then one row, say, the first, may by Theorem 2 be expressed as a linear combination of the others.

The illustration below, with a determinant of order 3 is easily extended to any order.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{21} + \ell a_{31} & ka_{22} + \ell a_{32} & ka_{23} + \ell a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

But, by Theorem 5 of Supplement A, this last determinant may be written as a sum of determinants, and equals

$$\begin{vmatrix} ka_{21} & ka_{22} & ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \ell a_{31} & \ell a_{32} & \ell a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

The application of Theorem 4 of Supplement A shows that both of these are equal to zero, and therefore, so is the original determinant.

Application to vectors.

THEOREM 4. Any set of vectors which includes the zero vector is L.D.

THEOREM 5. Two non-zero vectors are L.D. if and only if they are collinear.

(a) If $\overrightarrow{P}$ and $\overrightarrow{Q}$ are collinear, then, from Chapter 3, there exists a number k such that $\overrightarrow{P} = k\overrightarrow{Q}$. Therefore $1\overrightarrow{P} - k\overrightarrow{Q} = \overrightarrow{0}$, therefore $\overrightarrow{P}$ and $\overrightarrow{Q}$ are L.D.

(b) If $\overrightarrow{P}$ and $\overrightarrow{Q}$ are L.D. then there exist numbers a and b not both $= \overrightarrow{0}$, such that $a\overrightarrow{P} + b\overrightarrow{Q} = \overrightarrow{0}$. Suppose $a \neq 0$, then $\overrightarrow{P} = -\frac{b}{a}\overrightarrow{Q}$, that is $\overrightarrow{P} = k\overrightarrow{Q}$ which means that $\overrightarrow{P}$ and $\overrightarrow{Q}$ are collinear.

Corollary. $[p,q]$, $[r,s]$ are collinear if and only if $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 0$.

THEOREM 6. In the plane, any set of three non-zero vectors is L.D.

(a) If any two are collinear, they are L.D. and then so is the set of three.

(b) If no two are collinear, then, for any c, we will show that we can always find values for a and b such that

$$a\vec{P} + b\vec{Q} + c\vec{R} = \vec{0} ;$$

that is we can find a, b, for any p, q, r, s, t, u, c such that

$$a[p,q] + b[r,s] + c[t,u] = [0,0] .$$

This requires unique solutions for a, and b, in the equations

$$pa + rb = -ct$$
$$qa + sb = -cu .$$

But, from the hypothesis that $\vec{P}$ and $\vec{Q}$ are not collinear, we have $\begin{vmatrix} p & r \\ q & s \end{vmatrix} \neq 0$, and this is exactly the condition that there be unique solutions for a and b in the equations above.

Corollary. In the plane, any vector can be expressed as a linear combination of any pair of non-collinear vectors. That is, if $\vec{P}$ and $\vec{Q}$ are not collinear, then, for any $\vec{X}$ we can find numbers a and b so that $a\vec{P} + b\vec{Q} = \vec{X}$. (Compare with Theorem 3-5).

Terminology. If any vector of the plane can be expressed as a linear combination of the members of some set $S = \{\vec{P},\vec{Q},\vec{R},\ldots\}$, then S is said to span the plane. A set of vectors which is L.D. and which spans the plane is called a basis set, or simply a basis for the plane.

Note: (1) Any pair of non-collinear vectors forms a basis for the plane.

(2) These concepts generalize in a natural and interesting way to higher dimensions:
The set of vectors, $\{[1,0] \cdot [0,1]\}$ is what is called the "natural basis" for the plane, since, $[a,b] = a[1,0] + b[0,1]$.
The natural basis for three dimensions is the set $\{[1,0,0] , [0,1,0] , [0,0,1]\}$; etc.

(3) The number of vectors in the basis is the same as the dimension of the space. Thus, we may define a space of four dimensions as one in which there is at least one set of four L.I. vectors, but in which every set of five vectors is L.D. Similar definitions may be stated for five and higher dimensions.

Applications to Geometry

1. The lines $ax + by = c$, and $px + qy = r$ intersect in a point if and only if the corresponding equations have a unique solution for x and y, that is, if and only if $\begin{vmatrix} a & b \\ p & q \end{vmatrix} \neq 0$. This is true if and only if the left members of these equations are L.I. If the left members are L.D. then the lines will be parallel or coincident, as can easily be shown.

2. The concept introduced above generalizes easily. The planes:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

meet in a single point if and only if the left members of these equations are L.I. If they are L.D. then the planes may be related in various ways. All three may be parallel, two or three of them may coincide, two may be parallel and intersect the third, they may intersect in three parallel lines, etc. We leave the interested student to discover, either by his own research or by reference to other books, the connection between the dispositions of the planes, and the relations among the coefficients in their equations.

Supplement D

Supplements to Chapters 2, 3, and 8

POINTS, LINES, AND PLANES

The material in this supplement previously appeared as Chapter 4 in the preliminary edition. Parts of that chapter were retained in the text you are now using. These sections include significant material which may be of interest to you.

D-1. Choice of Methods

In this chapter we shall consider some questions about the undefined elements of geometry - points, lines, and planes. When do they intersect? How are they separated? What about betweenness? For answering these and other questions, we have developed the basic tools in the earlier chapters; it will be part of our task to select from among these tools those appropriate to the solution of a particular problem.

Sometimes we shall start with the general case and then take special cases. You may recall proving Desargues' Theorem in 3-space, and then showing that it holds in 2-space. At other times, we start with a more limited case and then generalize. Thus we considered distance first on a line, then in 2-space, and so on.

We have available different forms of representation. In a problem about a particular line, our representation of it may depend on what is known about it, what we want to prove about it, or other considerations. For example, if you are told that the x-intercept for a certain line is 2 and the y-intercept -3, you might choose as its equation $\frac{x}{2} + \frac{y}{-3} = 1$. If you are concerned with the amount of rotation of a line about a fixed point, you might want to use that point as pole of a polar coordinate system and write for the line $\theta = k$. A relation such as $r = \theta$, expressible most simply in polar coordinates, would be much more complicated to look at and to graph in rectangular coordinates. (You might want to try this.) In Chapter 4 vector methods are used to prove theorems of geometry that you proved earlier in other ways.

Our point here is that in this text from this point on you can expect to see a variety of representations and methods. In Sections D-2 and D-3, for example, rectangular coordinates and the equation $ax + by + c = 0$ for a line are chosen because it is desired to emphasize the relation of the geometric problem to an algebraic problem of solving systems of equations. In the same fashion, you have freedom to select the form of representation and the method that seems appropriate in a particular problem. Sometimes a few minutes spent first in deciding how to locate a coordinate system will save much time in solving a problem. Often there is no single simplest or best method.

D-2. Collinearity.

The geometric problem of whether three points are collinear corresponds to the algebraic problem of whether three pairs of values of two variables are solutions of the same linear equation in two variables.

Consider distinct points $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_3 = (x_3,y_3)$. Using the two-point form of the equation of a line derived in Section 2-5, the equation of the line $\overleftrightarrow{P_2P_3}$ can be written

$$y - y_3 = \frac{y_2 - y_3}{x_2 - x_3}(x - x_3) .$$

This we rewrite as

$$(y - y_3)(x_2 - x_3) = (y_2 - y_3)(x - x_3) .$$

If we multiply out and collect terms involving x and y, we have

(1) $$(y_2 - y_3)x - (x_2 - x_3)y + (x_2y_3 - x_3y_2) = 0 .$$

If we write the terms in parentheses as second order determinants (Appendix A), (1) becomes

$$x\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - y\begin{vmatrix} x_2 & 1 \\ x_3 & 1 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = 0 .$$

Using x , y , and 1 as the elements of the first row of a third order determinant, we can then write the equation in the form

$$(2) \qquad \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 .$$

Since (2) is an equation of the line P_2P_3 , the point P_1 is on this line if and only if

$$(3) \qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 .$$

Thus (3) is a compact form in which to write the condition that three points are collinear.

If three given points are not collinear, they determine a triangle. We choose a rectangular coordinate system so that the triangle is entirely in the first quadrant and name the points P_1 , P_2 , P_3 , in a counterclockwise order around the triangle, as shown in Figure D-1.

If the points P_1 , P_2 , P_3 are not collinear, they determine a triangle. To find its area we draw perpendiculars P_1F_1 , P_2F_2 , P_3F_3 to the x-axis. We can find the area K of $\Delta P_1P_2P_3$ by subtracting the area of trapezoid $F_1P_1P_3F_3$ from the sum of the areas of trapezoids $F_1P_1P_2F_2$ and $F_2P_2P_3F_3$.

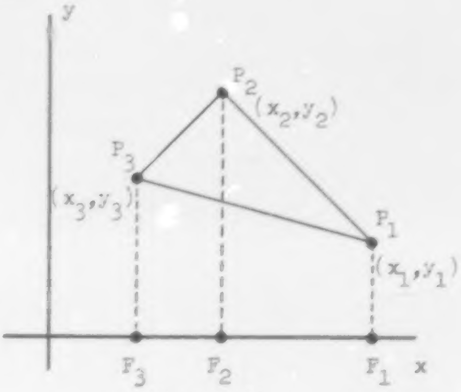


Figure D-1

$$K = \text{Area } F_1P_1P_2F_2 + \text{Area } F_2P_2P_3F_3 - \text{Area } F_1P_1P_3F_3 .$$

$$K = \frac{1}{2}(x_1 - x_2)(y_1 + y_2) + \frac{1}{2}(x_2 - x_3)(y_2 + y_3) - \frac{1}{2}(x_1 - x_3)(y_1 + y_3) ,$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1) ,$$

$$(4) \qquad K = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] ,$$

$$(5) \quad \text{or} \quad K = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| .$$

The student should verify that Equations (4) and (5) are equivalent. The value of the determinant in (5) will be positive if the vertices are named as in Figure D-1 so that traverse of the perimeter in the order $P_1P_2P_3$ is counterclockwise. If it is clockwise, the value of the determinant will be negative.

We notice that the determinant in (5) is the same as the one used to write (3), the condition that three points are collinear. This is not surprising, as it is intuitively obvious that three points are collinear if and only if the area of the "triangle" they determine is zero.

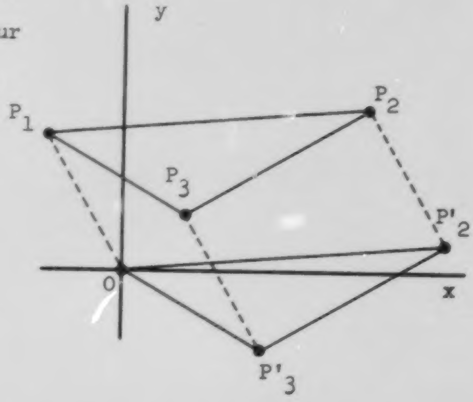
Formula (3) can be obtained in a different way by using vectors. In Section 3-8 we saw that the area of the triangle OXY, where $X = (x_1,x_2)$ and $Y = (y_1,y_2)$, is

$$K = \frac{1}{2}|x_1y_2 - x_2y_1| .$$

We use this result to find the area of an arbitrary triangle.

We name the vertices $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_3 = (x_3,y_3)$, so that our results shall have the same notation as the preceding development. We add the vector $-\vec{P}_1$ to each of the vectors $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$ to obtain the vectors $\vec{P}_1 - \vec{P}_1 = \vec{0}$, $\vec{P}_2 - \vec{P}_1 = \vec{P}_2{}'$, $\vec{P}_3 - \vec{P}_1 = \vec{P}_3{}'$ where

$$\vec{P}_2{}' = [x_2 - x_1 , y_2 - y_1]$$
$$\vec{P}_3{}' = [x_3 - x_1 , y_3 - y_1] .$$



Triangle $OP_2{}'P_3{}'$ is congruent to triangle $P_1P_2P_3$. Thus the area of triangle $P_1P_2P_3$ is

$$K = \frac{1}{2}\left|(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} 0 & 0 & 1 \\ x_2 - x_1 & y_2 - y_1 & 1 \\ x_3 - x_1 & y_3 - y_1 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

Exercises D-2

1. For each of the following find out whether the points whose coordinates are given are collinear; if not, find the area of the triangle that is determined.

   (a) $(7,0)$, $(4,-1)$, $(13,2)$
   (b) $(3,2)$, $(-2,-7)$, $(15,5)$
   (c) (a,b) , $(-a,-b)$, (c,d)
   (d) $(b,0)$, $(0,-b)$, $(a , a - b)$

2. Consider the triangle with vertices $P_1 = (0,0)$, $P_2 = (a,0)$, $P_3 = (b,c)$ and the value (not the absolute value) of the determinant in (5) . evaluate this determinant for P_1 , P_2 , P_3 . Evaluate it for $Q_1 = (0,0)$, $Q_2 = (b,c)$, $Q_3 = (a,0)$; for $R_1 = (a,0)$, $R_2 = (b,c)$, $R_3 = (0,0)$; and also for $S_1 = (b,c)$, $S_2 = (a,0)$, $S_3 = (0,0)$. Does the way you go around the triangle make a difference? Does the vertex at which you start make a difference? Try to state some general conclusions.

3. Prove that the area of the triangle with vertices $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_3 = (x_3,y_3)$ is

$$K = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right|$$

Note: The equation above may be written

$$K = \frac{1}{2}\left|\sum_{i=1}^{3}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix}\right| ,$$

where we interpret x_4 as x_1 and y_4 as y_1.

This generalizes immediately, giving the following formula for the area of a polygon with n vertices $P_i = (x_i,y_i)$:

$$K = \frac{1}{2}\left|\sum_{i=1}^{n}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix}\right| ,$$

where we interpret x_{n+1} as x_1 and y_{n+1} as y_1.

4. Find the area of the quadrilateral whose vertices are $P_1 = (4,1)$, $P_2 = (-1,3)$, $P_3 = (-3,-28)$, $P_4 = (2,-1)$, first by adding the areas of $\Delta P_1P_2P_3$ and $\Delta P_3P_4P_1$, and then by using the formula in Problem 3 above.

5. Prove that points $A = (-2,1)$, $B = (2,-2)$, and $C = (6,-5)$ are collinear.

   (a) Use condition (3) .
   (b) Show that $\vec{B} - \vec{A} = k(\vec{C} - \vec{A})$.
   (c) Show that $d(A,B) + d(B,C) = d(A,C)$.

D-3. Concurrence.

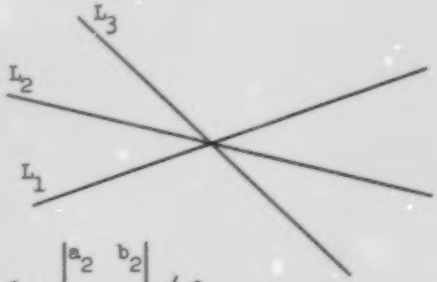
The geometric problem of whether three lines are concurrent corresponds to the algebraic problem of whether one pair of values of two variables satisfies three different linear equations in two variables.

We consider three lines L_1 , L_2 , and L_3 , with equations

$$(1)\qquad \begin{aligned} L_1 &: a_1x + b_1y + c_1 = 0 \\ L_2 &: a_2x + b_2y + c_2 = 0 \\ L_3 &: a_3x + b_3y + c_3 = 0 . \end{aligned}$$

These lines may be related in any one of the following ways; we shall consider the analytic conditions for each.

(a) The lines may be concurrent. This is the case of most interest to us since it represents the usual situation in which there is a unique solution of the three equations. The equations represent three distinct lines with one and only one point in common. For this, any two of the lines must intersect in a point, and that point must lie on the third line. From our study of Intermediate Mathematics we know that this first requirement means that we must have



$$(2) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 , \quad \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \neq 0 , \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \neq 0 .$$

The second condition requires that the intersection of, say, L_1 and L_2, must lie on L_3. If $P_1 = (x_1, y_1)$ represents the intersections of L_1 and L_2, we may write its coordinates

$$x_1 = \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} , \quad y_1 = \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} .$$

The condition that P_1 is on L_3 is

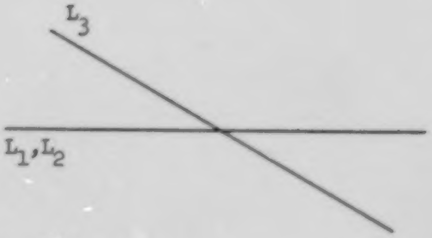
$$a_3 \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + b_3 \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + c_3 = 0 ,$$

which can be written more compactly as

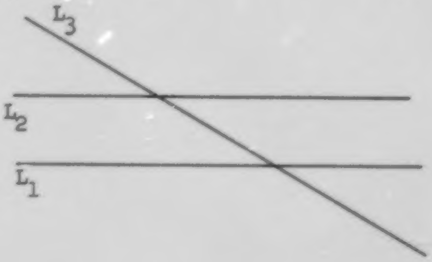
$$(3) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 .$$

Thus the condition that three distinct lines be concurrent is that the determinant of their coefficients is zero.

(b) Two lines may coincide and be intersected by the third line. In this case the third order determinant is zero and there is a unique solution of the three equations, but this case may be distinguished from (a) by noting that one of the determinants of (2) is zero.

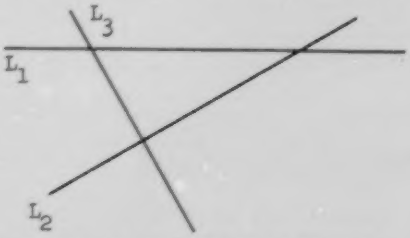


(c) Two lines may be parallel and be intersected by the third line. The student must be careful to distinguish this case from case (b), because here there is not a unique solution. This case resembles (b) in that one of the determinants of (2) is zero, but the determinant of the coefficients is not zero.



(d) The three lines may coincide. There is not a unique solution in this case since any solution of one equation is also a solution of each of the others. The third order determinant is zero as are all three determinants of (2). There are two other distinguishable cases which have these same algebraic conditions. The student may be interested in describing these cases and discovering how to distinguish them from case (d).

(e) Each line may intersect each of the others in a single point. Condition (2) holds, but the third order determinant is not zero. This is the case one is most likely to observe from three randomly chosen lines.



We might approach the question of concurrence in a somewhat different fashion. Let L_1 and L_2 be lines with equations given in (1). Then if m and n are any numbers not both equal to zero, the equation

$$m(a_1x + b_1y + c_1) + n(a_2x + b_2y + c_2) = 0 \tag{4}$$

is the equation of a line, since it is a first-degree equation in x and y.

If L_1 and L_2 intersect in $P_1 = (x_1, y_1)$, then (4) represents, for suitable choices of m and n , any line through P_1 . If L_1 and L_2 are parallel, then (4) represents, for suitable choices of m and n , any line parallel to L_1 and L_2 . If L_1 and L_2 coincide, then (4) represents that same line. Proof of these last statements will be left to the interested student.

Equation (4) represents what is often called a family of lines; that is, for suitable values of m and n it represents all the lines containing the intersection of L_1 and L_2 . Thus a condition that three distinct lines (with equations in the form $ax + by + c = 0$) be concurrent is that the left member of the equation of one of them is a linear combination of the left members of the equations of the other two.

Example 1. Find a value of k for which lines with the following equations will be concurrent. (Assume $k \neq -1$)

$$\begin{aligned} x - y &= 0 \\ 3x + 2 &= 0 \\ kx + y + 1 &= 0 \end{aligned}$$

Solution. We observe that the lines are not parallel (they satisfy condition (2)); we then use condition (3) .

$$\begin{vmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ k & 1 & 1 \end{vmatrix} = 0$$

We find that $k = \frac{1}{2}$.

Example 2.

(a) Find an equation that represents a line through the intersection of lines with equations $x + 3y - 3 = 0$ and $2x - 3y - 6 = 0$.

(b) Find an equation of the member of this family of lines

(1) that has slope equal to $\frac{3}{2}$.

(2) that contains the point $(0,3)$.

Solution.

(a) Using Equation (4) we write $m(x + 3y - 3) + n(2x - 3y - 6) = 0$, or $(m + 2n)x + (3m - 3n)y + (-3m - 6n) = 0$.

(b) (1) From the last equation in (a) we have an expression for the slope, which we set equal to $\frac{3}{2}$ and simplify.

$$- \frac{m + 2n}{3m - 3n} = \frac{3}{2}$$

$$- 2m - 4n = 9m - 9n$$

$$- 11m + 5n = 0$$

We let $m = 5$, $n = 11$, and substitute these values in the equation in (a) .

$$27x - 18y - 81 = 0$$

Or, more simply, $3x - 2y - 9 = 0$.

(2) If the line is to contain the point $(0,3)$, these coordinates must satisfy the first equation in (a) , therefore

$$m(0 + 9 - 3) + n(0 - 9 - 6) = 0 .$$

Simplifying, we have

$$6m - 15n = 0 .$$

We let $m = 5$, $n = 2$, and obtain

$$x + y - 3 = 0$$

as an equation of the desired line.

Exercises D-3

1. Are the lines with the given equations concurrent? If so, what is their common point?

   (a) $2x - 3y + 6 = 0$, $3x + 4y - 12 = 0$, $x - 4 = 0$
   (b) $x + y - 3 = 0$, $3x - y + 1 = 0$, $2x - 1 = 0$
   (c) $x - y = 4$, $y = x + 7$, $3x - 3y + 5 = 0$

2. For each of the following, determine a real number k such that the equations represent concurrent lines.

   (a) $x - 3y - 5 = 0$, $3x + y + 5 = 0$, $kx - 3y - 2 = 0$
   (b) $x + ky - 3 = 0$, $kx - 7y - 6 = 0$, $2x - y - 3k = 0$

3. Given lines L_1 , L_2 with equations $3x - 2y + 5 = 0$ and $x + 4y - 1 = 0$; write an equation that represents any line through the point of intersection of L_1 and L_2 . Then find the member of this family of lines that

   (a) has the slope $\frac{3}{4}$.

   (b) is perpendicular to L_1 .

   (c) contains the origin.

   (d) contains the point $(5,2)$.

   (e) has a y-intercept of 1 .

4. Find an equation of the line parallel to the line whose equation is $3x - y + 7 = 0$, and containing the point of intersection of the lines whose equations are $5x - y - 3 = 0$ and $x + y - 2 = 0$.

5. Given the triangle determined by points $A = (a,0)$, $B = (0,b)$, $C = (c,0)$.

   (a) Show that the medians are concurrent, and find their point of intersection. (This point is called the centroid. It was discussed and a vector proof of concurrency given in Example 2 , Section 3-8.)

   (b) Show that the altitudes are concurrent, and find their point of intersection. (This point is called the orthocenter.)

   (c) Show that the perpendicular bisectors of the sides are concurrent, and find their point of intersection. (This point is called the circumcenter; it is the center of the circumscribed circle of the triangle.)

   (d) Show that the centroid, the orthocenter, and the circumcenter of this triangle are collinear.

   (e) Do you think that what you have proved for triangle ABC is true for any triangle? Give reasons for your answer.

6. Prove that, in a trapezoid, the diagonals and the line drawn through the midpoints of the parallel sides meet in a point.

D-4. Intersections and Parallelism

If two sets have at least one member in common they are said to intersect. We consider in this section, points, lines and planes, and their possible intersections. If set S is a subset of set T , then their intersection is all of S , and we sometimes say that S lies on, or in, T , or S is embedded in T . Thus a point may lie on a line, or a line may be embedded in a plane. Our analytic representations of these sets makes it possible to develop simple criteria for many of these relationships.

Point and Point: P_1 , P_2 . This case is easy to analyze but a good place to start. Two points intersect if and only if they coincide. Their analytic representations are simply their coordinates, which must be identical or equivalent in accordance with the definition of equivalence given when the coordinate systems were introduced.

In rectangular coordinates $P = (3,5)$ differs from $P = (5,3)$. In polar coordinates $P = (6,\pi)$ is the same as $P = (-6,0)$ and $P = (6,3\pi)$.

Point and Line: P_1 , L . A point is on a line if and only if a set of coordinates of the point satisfies an equation of the line. The point $P_1 = (x_1,y_1)$ lies on the line $L : ax + by + c = f(x,y) = 0$, if and only if $f(x_1,y_1) = 0$. The point $P_1 = (x_1,y_1)$ lies on $L : x = a + \ell t$, $y = b + mt$, if and only if there is some value of t , say t_1 , such that $x_1 = a + \ell t_1$, and $y_1 = b + mt_1$. If P_1 and L had been given relative to a polar coordinate system, the discussion would require simple modifications, which are left to the student. The extension of the discussion to 3-space can also be made, with minor revisions which are also left to the student.

Examples.

(a) $P = (1,3)$ is on $L : 3x - 2y + 3 = 0$, because $3(1) - 2(3) + 3 = 0$.

(b) $P = (1,4)$ is not on $L : x = 3 + t$, $y = 2 - 3t$, because the equations $1 = 3 + t$, $4 = 2 - 3t$ impose contradictory conditions on t .

(c) $P = (12,60^\circ)$ is on $L : r = \frac{6}{\cos\theta}$, because $12 = \frac{6}{\cos 60^\circ}$.

Similarly, $Q = (6\sqrt{2},\frac{\pi}{4})$ and $R = (12,-60^\circ)$ are also on L .

(d) $P = (2,5,-1)$ is on $L : x = 3 + t$, $y = 2 - 3t$, $z = 1 + 2t$, since the equation $2 = 3 + t$ gives a value for t , namely $t = -1$, which is consistent with the equations: $5 = 2 - 3t$ and $-1 = 1 + 2t$.

__Point__ __and__ __Plane__: P_1 , M . The discussion is left to the student, who is referred to the paragraph above.

__Line__ __and__ __Line__: L_1 , L_2 . __2-space__. Two lines in the same plane may have (1) just one, or (2) all, or (3) no points in common. If the lines are $L_1 : a_1x + b_1y + c_1 = f_1(x,y) = 0$, and $L_2 : a_2x + b_2y + c_2 = f_2(x,y) = 0$, the analytic counterparts of these 3 cases are presented below. Proofs, which are not difficult, are left to the student.

(1) L_1 , L_2 intersect in just one point if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

(2) L_1 , L_2 coincide if and only if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0$$

Note that if any two of these determinants are equal to zero, so is the third. Note also, that if this condition is satisfied, there is a non-zero number, k , such that $f_1(x,y) = kf_2(x,y)$.

(3) L_1 , L_2 are parallel if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$,

__and__ either $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ or $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \neq 0$.

(a) Note that, if either of these is different from zero, so is the other.

(b) Note that, for any numbers p and q , the equation $pf_1(x,y) + qf_2(x,y) = 0$ is, in general, an equation of a line, L_3 . If L_1 , L_2 intersect, then L_3 will go through that point of intersection; if L_1 , L_2 coincide, then L_3 will coincide with them; and if L_1 and L_2 are parallel, then L_3 will be parallel to both of them.

If equations for L_1 and L_2 had been presented in parametric or vector form, then the analytic representations of the three cases above would have a somewhat different appearance. The development of these representations is called for in one of the exercises at the end of this section.

3-space. Two lines in 3-space may have (1) just one point in common, (2) all points in common, or no points in common. In 2-space, this last condition requires that the lines be parallel, but in 3-space, lines that have no point in common may be (3) parallel, if they lie in one plane, or (4) skew, if they do not.

The discussion of the first three cases is analogous to the corresponding discussion of the lines in 2-space, but the equations are more complicated.

Suppose L_1 goes through $P_1 = (a_1, b_1, c_1)$ with direction numbers (ℓ_1, m_1, n_1), and L_2 through $P_2 = (a_2, b_2, c_2)$ with direction numbers (ℓ_2, m_2, n_2). Therefore we have equations $L_1 : x = a_1 + \ell_1 s$, $y = b_1 + m_1 s$, $z = c_1 + n_1 s$; and $L_2 : x = a_2 + \ell_2 t$, $y = b_2 + m_2 t$, $z = c_2 + n_2 t$.

(1) If L_1, L_2 intersect at a unique point $P' = (x', y', z')$, there must be values of the parameters, say s' and t', such that

$$\begin{aligned} x' &= a_1 + \ell_1 s' = a_2 + \ell_2 t' \\ y' &= b_1 + m_1 s' = b_2 + m_2 t' \\ z' &= c_1 + n_1 s' = c_2 + n_2 t' . \end{aligned}$$

These are three linear equations in s' and t', which we may write:

$$\begin{aligned} \ell_1 s' - \ell_2 t' &= a_2 - a_1 \\ m_1 s' - m_2 t' &= b_2 - b_1 \\ n_1 s' - n_2 t' &= c_2 - c_1 . \end{aligned}$$

There is a unique common solution if and only if there is a unique solution to any two of these equations which will also satisfy the third. The solution, if any, for the first two equations, say, is:

$$s' = \frac{\begin{vmatrix} a_2 - a_1 & -\ell_2 \\ b_2 - b_1 & -m_2 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}}, \quad t' = \frac{\begin{vmatrix} \ell_1 & a_2 - a_1 \\ m_1 & b_2 - b_1 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}}$$

(Note that these solutions require $\begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} \neq 0$.) The corresponding requirements that there be unique solutions for any two of the above three equations are

$$\begin{vmatrix} \ell_1 & \ell_2 \\ n_1 & n_2 \end{vmatrix} \neq 0 \text{ , and } \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \neq 0 .$$

If the s' , t' values found above are substituted in the third equation we have:

$$\frac{n_1\begin{vmatrix} a_2 - a_1 & -\ell_2 \\ b_2 - b_1 & -m_2 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}} - \frac{n_2\begin{vmatrix} \ell_1 & a_2 - a_1 \\ m_1 & b_2 - b_1 \end{vmatrix}}{\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix}} = c_2 - c_1 ;$$

therefore,

$$n_1\begin{vmatrix} a_2 - a_1 & -\ell_2 \\ b_2 - b_1 & -m_2 \end{vmatrix} - n_2\begin{vmatrix} \ell_1 & a_2 - a_1 \\ m_1 & b_2 - b_1 \end{vmatrix} - (c_2 - c_1)\begin{vmatrix} \ell_1 & -\ell_2 \\ m_1 & -m_2 \end{vmatrix} = 0 .$$

This may, after some algebraic juggling, be written in the form

$$(a_2 - a_1)(m_1n_2 - m_2n_1) - (b_2 - b_1)(\ell_1n_2 - \ell_2n_1) + (c_2 - c_1)(\ell_1m_2 - \ell_2m_1) = 0 ;$$

and this in turn may be written in determinant form:

$$\Delta = \begin{vmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 .$$

Note that the elements of the rows are direction numbers for $\overleftrightarrow{P_1P_2}$, L_1 , L_2 .

(2) , (3) If L_1 and L_2 are parallel or coincident, their direction numbers are equivalent, and all the second order minors of the last two rows must equal zero, and therefore Δ must equal zero. If L_1 and L_2 coincide, they coincide also with $\overleftrightarrow{P_1P_2}$, whose direction numbers must be equivalent to those of L_1 and L_2 , and in that case all the second order minors of Δ must equal zero. If L_1 and L_2 are parallel, then they both intersect $\overleftrightarrow{P_1P_2}$ whose direction numbers may not be equivalent to those of L_1 and L_2 , and in that case the second order minors of Δ which include members from the first row may not all equal zero.

(4) Finally, if L_1 and L_2 are skew , $\Delta \neq 0$.

Example. Consider the lines

$$L_1 : x = 2 + 3t \ , \ y = 3 - t \ , \ z = 4 + 5t \ ,$$

$$L_2 : x = 2 + 2t \ , \ y = -1 + t \ , \ z = 0 + 3t \ ,$$

$$L_3 : x = 3 + 6t \ , \ y = 2 - 2t \ , \ z = 1 + 10t \ ,$$

$$L_4 : x = -1 + 9t \ , \ y = 4 - 3t \ , \ z = -1 + 15t \ .$$

(a) For L_1 and L_2 , $\Delta = \begin{vmatrix} 0 & -4 & -4 \\ 3 & -1 & 5 \\ 2 & 1 & 3 \end{vmatrix} = -24 \neq 0$. $\therefore$ L_1 and L_2 are skew.

(b) For L_1 and L_3 , $\Delta = \begin{vmatrix} 1 & -1 & -3 \\ 3 & -1 & 5 \\ 6 & -2 & 10 \end{vmatrix} = 0$. $\therefore$ L_1 and L_3 are not skew but may intersect in just one point or be parallel or coincident. However,

$\begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 6 & 10 \end{vmatrix} = \begin{vmatrix} -1 & 5 \\ -2 & 10 \end{vmatrix} = 0$, $\therefore$ L_1 and L_3 cannot intersect in just one point, but must be parallel or coincident. Coincidence requires all second order minors of Δ to equal zero, and, since

$$\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = 2 \neq 0 \text{ , the lines are not coincident}$$

and must be parallel.

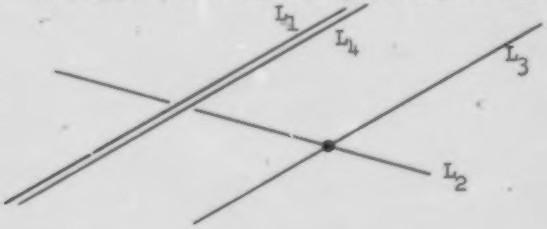
(c) For L_1 and L_4 , $\Delta = \begin{vmatrix} -3 & 1 & -5 \\ 3 & -1 & 5 \\ 9 & -3 & 15 \end{vmatrix} = 0$, and also all the second order minors of Δ equal zero. Therefore L_1 and L_4 coincide.

(d) For L_2 and L_3 , $\Delta = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \\ 6 & -2 & 10 \end{vmatrix} = 0$, $\therefore L_2$ and L_3 are not skew, but may intersect in just one point, or be parallel or coincident. These last two possibilities are eliminated by the fact that

$$\begin{vmatrix} 2 & 1 \\ 6 & -2 \end{vmatrix} = -10 \neq 0 \text{ , and } \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 \neq 0 \text{ .}$$

Therefore L_2 and L_3 intersect in just one point, which can be found by the methods in the section above to be $P(6,1,6)$.

The sketch below suggests the relative positions of the four lines.



Exercise. Show analytically that

(a) L_2 and L_4 are skew.

(b) L_3 and L_4 are parallel.

Line and Plane: L , M . A line may (1) be parallel to a plane, (2) be embedded in a plane, or (3) intersect a plane in just one point. In this last case we sometimes say that the line pierces the plane. We develop the analytic counterparts of these three cases.

(1) Suppose L goes through $P_0 = (a_0,b_0,c_0)$ with direction numbers (ℓ_0,m_0,n_0) ; then equations for L are $L : x = a_0 + \ell_0 t$, $y = b_0 + m_0 t$, $z = c_0 + n_0 t$. Suppose we have the plane

$M : px + qy + rz + s = f(x,y,z) = 0$.

Then L will be parallel to M if and only if no point of L lies in M, that is, if there is no value of t such that $p(a_0 + \ell_0 t) + q(b_0 + m_0 t) + r(c_0 + n_0 t) + s = 0$. This is an equation in t , which may be written

$$(pa_0 + qb_0 + rc_0 + s) + (p\ell_0 + qm_0 + rn_0)t = 0 .$$

The coefficient of t resembles the algebraic form of the inner product of two vectors. (See Section 3-5) It is convenient to borrow the algebraic symbolism of vectors and represent this coefficient as the "inner product" of the "vectors" $[p,q,r]$ and $[\ell_0,m_0,n_0]$. With this symbolism, the above equation becomes,

$$f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0,m_0,n_0]t = 0 .$$

For this linear equation in t to have no solution, it is necessary and sufficient that both: $f(a_0,b_0,c_0) \neq 0$, and $[p,q,r] \cdot [\ell_0,m_0,n_0] = 0$, which are the conditions for L to be parallel to M . These may be recognized as requiring that P_0 , which is a point of L , not lie in M ; and that L be perpendicular to a normal line of M , as established earlier.

Example. Show that $L : x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$, is parallel to $M : 3x + 3y - z - 5 = f(x,y,z) = 0$.

Solution. The criteria developed in the text are satisfied, since:

(1) $f(3,4,1) = 9 + 12 - 1 - 5 = 15 \neq 0$, and

(2) $[2,-1,3] \cdot [3,3,-1] = 2(3) - 1(3) + 3(-1) = 6 - 3 - 3 = 0$.

We might also substitute, in the equation of M , the expressions for x , y , z as functions of t , and get $3(3 + 2t) + 3(4 - t) - (1 + 3t) - 5 = 0$, which leads to the contradiction $15 = 0$. Therefore L doesn't intersect M .

The x-axis, or any line parallel to it, has equations: $x = a_0 + \ell_0 t$, $y = b_0$, $z = c_0$, with direction numbers $(\ell_0, 0, 0)$. If a plane has an equation such as $M : qy + rz + s = 0$, its normal lines have direction numbers $(0,q,r)$. $\therefore$ M is parallel to the x-axis or contains it, since $[\ell_0, 0, 0] \cdot [0,q,r] = 0$.

In the same way, if a plane has an equation in general form in which the y term is missing, then the plane is parallel to, or contains the y-axis, and so on.

(2) If a line is embedded in a plane, then coordinates of every point of the line must satisfy an equation of the plane. If L and M are given as before: $L : x = a_0 + \ell_0 t$, $y =$ [illegible] $m_0 t$, $z = c_0 + n_0 t$, and $M : px + qy + rz + s = f(x,y,z) =$ [illegible] n this requirement is met if, for all t, $p(a_0 + \ell_0 t) + q(b_0 + m_0 t) + r(c_0 + n_0 t) + s = 0$. This may be written as: $(pa_0 + qb_0 + rc_0 + s) + (p\ell_0 + qm_0 + rn_0)t = 0$, or as: $f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0, m_0, n_0]t = 0$.

If this expression is to equal zero for all values of t then we must have: $f(a_0,b_0,c_0) = 0$ and $[p,q,r] \cdot [\ell_0, m_0, n_0] = 0$.

These conditions for embedding may be recognized as requiring that $P_0 = (a_0,b_0,c_0)$, which is a point of L, also be a point of M; and also that L, with direction number (ℓ_0, m_0, n_0) be perpendicular to a normal to M. We have previously used the fact that such a normal has direction numbers (p,q,r).

Example. Show that $L : x = 3 + 2t$, $y = 1 + t$, $z = 3 - t$, lies wholly in $M : 2x - 3y + z - 6 = f(x,y,z) = 0$.

Solution. Both conditions in the section above are satisfied, since

(a) the point $(3,1,3)$ is on M, since $f(3,1,3) = 0$, and

(b) a normal to M has direction numbers $(2,-3,1)$; and L is perpendicular to such a normal, since $[2,-3,1] \cdot [2,1,-1] = 4 - 3 - 1 = 0$.

(3) If we suppose L and M given as in the two cases above, then, if L intersects M in just one point, there must be a unique value of t, say t', such that $P' = (x',y',z')$ on L is also on M. That is, if $x' = a_0 + \ell_0 t'$, $y' = b_0 + m_0 t'$, $z' = c_0 + n_0 t'$, then $p(a_0 + \ell_0 t') + q(b_0 + m_0 t') + r(c_0 + n_0 t') + s = 0$. This is a linear equation in t' which may be written:

$$(pa_0 + qb_0 + rc_0 + s) + (p\ell_0 + qm_0 + rn_0)t' = 0 \text{ , or}$$

$$f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0,m_0,n_0]t' = 0 \text{ .}$$

A unique solution will exist if and only if the coefficient of t' is different from zero, that is, $[p,q,r] \cdot [\ell_0,m_0,n_0] \neq 0$. If this condition is satisfied, we may find the unique value of t':

$$t' = -\frac{f(a_0,b_0,c_0)}{[p,q,r] \cdot [\ell_0,m_0,n_0]} \text{ .}$$

With this value of t' we find the coordinates of P', the unique point of intersection of L and M.

Example. Find the point in which $L : x = 3 + 2t$, $y = 2 - 3t$, $z = 1 + t$, intersects $M : 2x - 3y + 4z - 5 = f(x,y,z) = 0$.

Solution. Either by direct substitution of expressions for x, y, z, in equations of L into the equation of M, or by application of the formula above, we obtain:

$$t' = -\frac{f(3,2,1)}{[2,-3,4] \cdot [2,-3,1]} = -\frac{2(3) - 3(2) + 4(1) - 5}{2(2) - 3(-3) + 4(1)}$$

$$t' = -\left(\frac{-1}{17}\right) = \frac{1}{17} \text{ . } \therefore \; P' = \left(\frac{53}{17}, \frac{31}{17}, \frac{18}{17}\right) \text{ .}$$

We may summarize the development in this section so far by observing that much of the analysis depends on the possiblity and nature of the solution of $f(a_0,b_0,c_0) + [p,q,r] \cdot [\ell_0,m_0,n_0]t = 0$. We exhibit the results of our analysis in the table below.

Case	$f(a_0,b_0,c_0)$	$[p,q,r] \cdot [\ell_0,m_0,n_0]$	numbers of solutions for t
(1) L_1 is parallel to M	$\neq 0$	$= 0$	none
(2) L_1 is embedded in M	$= 0$	$= 0$	infinitely many
(3) L_1 pierces M	any value	$\neq 0$	one

A significant problem, related to the problem of finding the distance between two skew lines, is to find parallel planes which contain two skew lines. Suppose the lines are $L_1 : x = a_1 + \ell_1 t_1$, $y = b_1 + m_1 t_1$, $z = c_1 + n_1 t_1$; and $L_2 : x = a_2 + \ell_2 t_2$, $y = b_2 + m_2 t_2$, $z = c_2 + n_2 t_2$. If the planes M_1 and M_2 are to be parallel, their normals must have equivalent direction numbers, and we may write their equations,

$M_1 : px + qy + rz + s_1 = f_1(x,y,z) = 0$; and

$M_2 : px + qy + rz + s_2 = f_2(x,y,z) = 0$. The problem is to find p , q , r , s_1 , and s_2 in terms of the constants which give us L_1 and L_2 , under the conditions imposed by the problem. Since L_1 and L_2 are embedded respectively in M_1 and M_2 , we have from the previous section ,

$f_1(a_1,b_1,c_1) = f_2(a_2,b_2,c_2) = 0$, and also

$[p,q,r] \cdot [\ell_1,m_1,n_1] = [p,q,r] \cdot [\ell_2,m_2,n_2] = 0$. These four equations are not sufficient to find the five values p , q , r , s_1 and s_2 , but we recognize that direction numbers need not be found uniquely; any equivalent set will do as well, to write equations for M_1 and M_2 . We assume that not all of (p,q,r) equal zero, and, in particular that, say, $r \neq 0$, in which case we have an equivalent set $(\frac{p}{r},\frac{q}{r},1)$; and the algebraic problem of solving four equations in four variables.

The algebraic conditions for solvability have their geometric counterparts, corresponding to the relative positions of L_1 and L_2 . We consider here only the situation in which L_1 and L_2 are skew. The general algebraic treatment of this case involves extensive algebraic manipulation, which we shall not go through. We will carry through the details in an example.

Example. Find parallel planes M_1 and M_2 which contain the lines $L_1 : x = 3 - t_1$, $y = 2 + 3t_1$, $z = 1 + 2t_1$; and $L_2 : x = -2 + 3t_2$, $y = 3 + 2t_2$, $z = 1 - 2t_2$.

Solution.

(1) L_1 is not parallel to L_2 , because their direction numbers are not equivalent.

(2) L_1 and L_2 do not meet, because the assumption of a common point imposes contradictory conditions on t_1 and t_2 . If we try to solve the system

$$\begin{cases} 3 - t_1 = -2 + 3t_2 \\ 2 + 3t_1 = 3 + 2t_2 \\ 1 + 2t_1 = 1 - 2t_2 , \end{cases}$$

the last two equations require $t_1 = \frac{1}{5}$, $t_2 = -\frac{1}{5}$, and these do not satisfy the first equation.

(3) Therefore, L_1 and L_2 are skew. Then, as in the section above, we consider planes $M_1 : px + qy + rz + s_1 = f_1(x,y,z) = 0$, and $M_2 : px + qy + rz + s_2 = f_2(x,y,z) = 0$. The conditions that L_1 and L_2 be perpendicular to a common normal N to planes M_1 and M_2 , become:

$$\begin{cases} (-1)p + 3(q) + 2(r) = 0 \\ (3)p + 2(q) - 2(r) = 0 . \end{cases}$$

We may rewrite these as

$$\begin{cases} -1(\frac{p}{r}) + 3(\frac{q}{r}) + 2 = 0 \\ 3(\frac{p}{r}) + 2(\frac{q}{r}) - 2 = 0 , \end{cases}$$

and these yield, by elementary methods, the solutions, $\frac{p}{r} = \frac{10}{11}$, $\frac{q}{r} = \frac{-4}{11}$. We may therefore use either the direction numbers $(\frac{10}{11}, \frac{-4}{11}, 1)$ or the equivalent $(10,-4,11)$. With these values of

p, q, r, we find s_1 and s_2 easily from the conditions that M_1 and M_2 contain points $P_1 = (3,2,1)$ and $P_2 = (-2,3,1)$ of L_1 and L_2 respectively, i.e.

$$p(3) + q(2) + r(1) + s_1 = 0 \, , \quad \therefore \; s_1 = -33 \, ,$$
$$p(-2) + q(3) + r(1) + s_2 = 0 \, , \quad \therefore \; s_2 = 21 \, .$$

Finally we have the equations of the planes

$M_1 : 10x - 4y + 11z - 33 = 0$; $M_2 : 10x - 4y + 11z + 21 = 0$.

Two Planes: M_1, M_2. Suppose these planes have respective equations:

$$M_1 : p_1x + q_1y + r_1z + s_1 = f_1(x,y,z) = 0 \, ,$$
$$M_2 : p_2x + q_2y + r_2z + s_2 = f_2(x,y,z) = 0 \, .$$

The planes may (1) coincide, (2) be parallel, or (3) intersect.

(1) The planes coincide if and only if every point of one of them is a point of the other, and this will be the case if and only if there is some non-zero number k such that $f_1(x,y,z) = kf_2(x,y,z)$, as may be easily seen.

(2) The planes will be parallel if and only if they have a common normal, but no common point. These conditions will both be met if there is a number $k \neq 0$, such that $p_1 = kp_2$, $q_1 = kq_2$, $r_1 = kr_2$ but $s_1 \neq ks_2$. The proof that this is so is left to the student.

(3) If two distinct planes intersect in a point $P_0 = (x_0,y_0,z_0)$, one of the earlier postulates of geometry requires that they intersect in a line containing P_0. We show, from the analytic representation and condition that this is so, and find the line, given the planes.

The general treatment would involve tedious computation, and would probably not be as enlightening as a specific example.

Example. Suppose two planes, $M_1 : 2x - 3y + z - 4 = f_1(x,y,z) = 0$, and $M_2 : x + 2y - 4z - 1 = f_2(x,y,z) = 0$, have the point $P_0 = (3,1,1)$ in common. Show that they have in common a line containing P_0.

Solution. If p and q are numbers not both zero, the equation $pf_1(x,y,z) + qf_2(x,y,z) = 0$ is, in general, an equation of a plane containing P_0 . This equation may be written as:

$$(2p + q)x + (-3p + 2q)y + (p - 4q)z + (-4p - q) = 0 .$$

If, in particular, $p = 1$, $q = -2$, this equation becomes $-7y + 9z - 2 = 0$ or $7y - 9z + 2 = 0$. The locus, in 3-space of this equation is, as shown in the previous section, a plane, parallel to the x-axis. Note that this plane contains $P_0 = (3,1,1)$, since $7(1) - 9(1) + 2 = 0$. If we subtract corresponding members of these two equations we get, as another equation of this plane, $7(y - 1) - 9(z - 1) = 0$.

In the same way, by taking $p = 2$, $q = 3$, we get the equation $7x - 10z - 11 = 0$, which represents a plane parallel to the y-axis, and also containing $P_0 = (3,1,1)$, since $7(3) - 10(1) - 11 = 0$. If we subtract corresponding members of these two equations we get $7(x - 3) - 10(z - 1) = 0$. These equations of the two planes parallel to the x- and y-axes, respectively, may be written:

$$\frac{y - 1}{9} = \frac{z - 1}{7} ,$$

$$\frac{x - 3}{10} = \frac{z - 1}{7} .$$

Note that these three fractional expressions are all equal and can be set equal to some common value t , from which we get $x = 3 + 10t$, $y = 1 + 9t$, and $z = 1 + 7t$.

These are clearly a set of parametric equations for a line L containing the point $(3,1,1)$. To show that L lies wholly in M_1 we must show, that for all values of t ,

$$2(3 + 10t) - 3(1 + 9t) + 1(1 + 7t) - 4 = 0 ,$$

that is,

$$6 + 20t - 3 - 27t + 1 + 7t - 4 = 0 ,$$

which becomes, for all t , $0 = 0$.

In the same way, to show that L lies wholly in M_2 , we must show, that for all values of t ,

$$1(3 + 10t) + 2(1 + 9t) - 4(1 + 7t) - 1 = 0 ,$$

that is,

$$3 + 10t + 2 + 18t - 4 - 28t - 1 = 0 ,$$

and, for all t , this becomes, $0 = 0$.

Exercises D-4

Consider the four lines given by the equations below for Exercises 1 to 6.

$$L_1 : \begin{cases} x = -2 + 3t_1 \\ y = 3 - t_1 \\ z = 4 - 2t_1 \end{cases} \quad L_2 : \begin{cases} x = 3 - 6t_2 \\ y = -5 + 2t_2 \\ z = 1 + 4t_2 \end{cases} \quad L_3 : \begin{cases} x = -5 + 3t_3 \\ y = 6 - 2t_3 \\ z = 13 - 8t_3 \end{cases}$$

$$L_4 : \begin{cases} x = 7 - 3t_4 \\ y = -6 + 4t_4 \\ z = 9 - 6t_4 \end{cases}$$

1. Determine for each pair below if the lines (a) intersect in just one point, or (b) are parallel, or (c) are coincident, or (d) are skew. If a pair intersect in just one point, find that point.

   (a) L_1 , L_2 (d) L_2 , L_3

   (b) L_1 , L_3 (e) L_2 , L_4

   (c) L_1 , L_4 (f) L_3 , L_4

2. Write an equation for the line which contains $P = (1,2,3)$ and is parallel to

   (a) L_1 . (c) L_3 .

   (b) L_2 . (d) L_4 .

3. Write equations of parallel planes M_1 and M_2 which contain respectively

   (a) L_1 and L_3 . (b) L_2 and L_4 .

4. Write an equation of a plane which

   (a) contains L_1 and is parallel to L_3 .

   (b) contains L_4 and is parallel to L_1 .

5. Write an equation for the plane which contains the origin and

   (a) L_1 . (c) L_3 .

   (b) L_2 . (d) L_4 .

L_A is said to *go over* L_B if L_A and L_B are disjoint (have no point in common), and there is a point P_A on L_A which is *above* a point P_B on L_B ; that is, such that $x_A = x_B$, $y_A = y_B$ and $z_A > z_B$. There is a corresponding definition for a line to go under another line. We show that L_1 goes over L_3 because if $x_1 = x_3$ and $y_1 = y_3$ we have $-2 + 3t_1 = -5 + 3t_3$, and $3 - t_1 = 6 - 2t_3$, therefore $t_1 = 1$ and $t_3 = 2$. For these values of t_1 and t_3 we have $z_1 = 2$ and $z_3 = -3$, $z_1 > z_3$, and therefore L_1 goes over L_3.

6. Determine the over or under relationship for these pairs of lines:

   (a) L_1 and L_4 (c) L_2 and L_4

   (b) L_2 and L_3 (d) L_3 and L_4

7. If L_A goes over L_B, and L_B goes over L_C, is it always, sometimes, or never true that L_A goes over L_C?

8. True or false? One of two disjoint lines is over the other. Explain.

Consider the four planes $M_1 : 3x - 2y + z - 5 = 0$, for Exercises 9 to 12. $M_2 : 2x + y - 3z + 4 = 0$; $M_3 : x + 3y - 2z - 1 = 0$, and $M_4 : -2x + y + 2z + 3 = 0$.

9. Find in parametric form, equations of the line of intersection of

   (a) M_1, M_2. (d) M_2, M_3.

   (b) M_1, M_3. (e) M_2, M_4.

   (c) M_1, M_4. (f) M_3, M_4.

10. Find the common intersection point, if any, of

    (a) M_1, M_2, M_3. (c) M_1, M_3, M_4.

    (b) M_1, M_2, M_4. (d) M_2, M_3, M_4.

    Note that we may use the results of Exercise 9 to facilitate the computation in Exercise 10.

11. Write an equation of the plane which contains the origin, and is parallel to:

    (a) M_1 (c) M_3

    (b) M_2 (d) M_4

12. (Refer to the lines at the top of this group of exercises.)
Find the point, if any, in which

(a) L_1 meets M_1 .   (c) L_3 meets M_3 .

(b) L_2 meets M_2 .   (d) L_4 meets M_4 .

13. Suppose equations of two lines in 2-space are given in parametric form. Develop criteria, in terms of the constants in these equations, for the various geometric relationships that may exist between the lines, as in Section 4-6D , where the equations were given in general form.

D-5. Perpendicularity and Angles between Lines and Planes

We have used quite freely in this chapter the definitions and tests for perpendicularity that had been developed in Chapter 2 . For the purposes of this chapter we consider angles between lines and planes in general, and perpendicularity as the special relationship that exists when these angles are right angles. We recall that an angle has been defined as the union of two non-collinear rays with a common end-point.

Two lines: L_1 , L_2 . We do not define angles between parallel or coincident lines. Although there may be some value in the consideration of "straight angles", or "zero angles", we feel that there is not sufficient application of these concepts in this text to warrant the time and effort that their treatment would entail. We have already developed in earlier sections analytic criteria to distinguish cases of parallelism or coincidence.

If L_1 and L_2 are neither parallel nor coincident we define the angles between them to be the angles formed by lines L'_1 and L'_2 which contain some common point, say, the origin, and are respectively coincident with or parallel to L_1 and L_2 . Note that this definition covers any intersecting or skew lines. Such lines determine four angles, which can be analytically distinguished only if there is some way of establishing, implicitly or explicitly, a sense on L_1 and L_2 .

2-space: Consider the intersecting lines $L_1 : x = a_1 + \lambda_1 t$, $y = b_1 + \mu_1 t$, and $L_2 : x = a_2 + \lambda_2 t$, $y = b_2 + \mu_2 t$, where λ_1 , μ_1 , λ_2 , μ_2 , are direction cosines. Then the lines L'_1 and L'_2 which go through the origin and are respectively parallel to or coincident with L_1 and L_2 have the equations:

$$L'_1 : x = \lambda_1 t \text{ , } y = \mu_1 t \text{ ; } \quad L'_2 : x = \lambda_2 t \text{ , } y = \mu_2 t \text{ .}$$

Note that λ_1 , μ_1 establish a sense along L_1 and L'_1 ; the "positive" part containing points for which $t > 0$; and so on. If, on L'_1 and L'_2 , we take $t = 1$, we get the points $P_1 = (\lambda_1, \mu_1)$ and $P_2 = (\lambda_2, \mu_2)$ on the positive rays $\overrightarrow{OP}_1$, $\overrightarrow{OP}_2$. We define the angle between L_1 and L_2 as given above, to be the angle formed by $\overrightarrow{OP}_1$ and $\overrightarrow{OP}_2$, which we designate as θ . Note that if we had taken for L_1 the equivalent direction cosines $-\lambda_1$, $-\mu_1$, these would have been established on L_1 a sense opposite to the original, and in that case the angle between L_1 and L_2 would have been the supplement of θ . It is not difficult to see that, for any choices of equivalent direction cosines for L_1 and L_2 the angle between L_1 and L_2 would be congruent either to θ or its supplement. These are the angles we mean when we speak of the angles formed by two lines.

From $\triangle OP_1P_2$ and the law of cosines we get

$d^2(P_1,P_2) = d^2(O,P_1) + d^2(O,P_2) - 2d(O,P_1)d(O,P_2) \cos \theta$. Note that

$d(O,P_1) = d(O,P_2) = 1$, and $d^2(P_1,P_2) = (\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)^2$

$$= \lambda_1{}^2 - 2\lambda_1\lambda_2 + \lambda_2{}^2 + \mu_1{}^2 - 2\mu_1\mu_2 + \mu_2{}^2$$

$$= 2 - 2\lambda_1\lambda_2 - 2\mu_1\mu_2 \text{ .}$$

Therefore $\quad 2 - 2\lambda_1\lambda_2 - 2\mu_1\mu_2 = 2 - 2 \cos \theta$

(1) and $\quad \text{Cos } \theta = \lambda_1\lambda_2 + \mu_1\mu_2$.

This is an unambiguous determination for one of the angles between L_1 and L_2 , namely that between the positive rays on L_1 and L_2 determined by the given direction cosines and $t > 0$. Another of the angles between L_1 and L_2 is clearly the supplement of θ .

Note that $L_1 \perp L_2$ if and only if the angles between them are right angles, that is, if and only if $\lambda_1\lambda_2 + \mu_1\mu_2 = 0$. This is a familiar criterion for perpendicularity.

We may indicate the corresponding results using direction numbers, rather than direction cosines. Note that when we set

$$\lambda = \frac{\ell}{\pm\sqrt{\ell^2 + m^2}}, \quad \mu = \frac{m}{\pm\sqrt{\ell^2 + m^2}}$$

there is an ambiguity introduced with the choice of sign for the radical. A particular pair of direction numbers entails an implicit sensing of the line, as with the case of direction cosines; the positive sign for both radicals preserves the original sensing. In terms of direction numbers, Equation (1) becomes

$$\text{Cos } \theta = \frac{\ell_1\ell_2 + m_1m_2}{\sqrt{\ell_1^2 + m_1^2}\,\sqrt{\ell_2^2 + m_2^2}}, \tag{2}$$

and the corresponding condition for perpendicularity becomes

$$\ell_1\ell_2 + m_1m_2 = 0 .$$

The development here resembles, as it should, the corresponding development with vectors, given in Section 3-7. We may, in these formulas, use the symbolism of vectors, to simplify their representations. We recognize that the vector $\overrightarrow{OP}_1 = [\lambda_1, \mu_1]$ and $\overrightarrow{OP}_2 = [\lambda_2, \mu_2]$. Therefore we may write Equation (1) in vector form:

$$\text{Cos } \theta = [\lambda_1, \mu_1] \cdot [\lambda_2, \mu_2] = \overrightarrow{OP}_1 \cdot \overrightarrow{OP}_2 .$$

In the same way, although we have not used vectors whose components are direction numbers, we may extend our symbolism and treat the expression $[\ell, m]$ algebraically as if it were a vector, in which case we may write Equation (2) in "vector" form:

$$\text{Cos } \theta = \frac{[\ell_1, m_1] \cdot [\ell_2, m_2]}{\|[\ell_1, m_1]\|\,\|[\ell_2, m_2]\|},$$

and the corresponding condition for perpendicularity as

$$[\ell_1, m_1] \cdot [\ell_2, m_2] = 0 .$$

Example 1. Find the angle between $L_1 : x = 2 + 3t\ ,\ y = 4 - t$, and $L_2 : x = 3 + t\ ,\ y = 2 + 2t$.

Solution.
$$\operatorname{Cos}\theta = \frac{(3)(1) + (-1)(2)}{\sqrt{3^2 + (-1)^2}\ \sqrt{1^2 + 2^2}}$$

$$\operatorname{Cos}\theta = \frac{3 - 2}{\sqrt{10}\ \sqrt{5}} = \frac{1}{\sqrt{50}} \approx .14$$

$$\therefore\quad \theta \approx 82^{\circ}$$

Example 2. Show that $L_3 : x = 3 - 5t\ ,\ y = 2 + 3t$ is perpendicular to $L_4 : x = 1 + 3t\ ,\ y = 4 + 5t$.

Solution. $(-5)(3) + (3)(5) = 0$, $\quad \therefore\ L_3 \perp L_4$.

Example 3. Find the angles between L_1 and L_2 , where L_1 contains the points $(3,4)$, $(-1,-1)$: and L_2 contains the points $(-4,6)$, $(3,0)$.

Solution. Since no sense is imposed on L_1 and L_2 we will find their angles of intersection.

We may take as direction numbers for L_1 , $(4,5)$ and for L_2 , $(-7,6)$. (Why?) Therefore:

$$\operatorname{Cos}\theta = \frac{(4)(-7) + (5)(6)}{\sqrt{4^2 + 5^2}\ \sqrt{(-7)^2 + 6^2}} \approx .034$$

$$\therefore\quad \theta \approx 88^{\circ}\ .$$

We may, most simply, find the other angle of intersection as the supplement of θ , but it is instructive to use equivalent direction numbers for L_1 which have the effect of reversing the sense induced by the first choice. We use now $(-4,-5)$, and $(-7,6)$ as pairs of direction numbers and get

$$\operatorname{Cos}\theta' = \frac{(-4)(-7) + (-5)(6)}{\sqrt{(-4)^2 + (-5)^2}\ \sqrt{(-7)^2 + 6^2}} \approx -\ .034$$

$$\therefore\quad \theta' \approx 92^{\circ}$$

which is, as we expected, supplementary to θ .

Example 4. Find the line L_5, to contain the point $(1,2)$ and be perpendicular to $L_1 : x = 2 + 3t$, $y = 4 - t$.

Solution. Suppose L_5 meets L_1 at $P = (a,b)$. Then we take direction numbers for L_5 as $(a - 1, b - 2)$. From the perpendicularity relationship we have $3(a - 1) - 1(b - 2) = 0$. From the fact that $P = (a,b)$ is on L_1, we have $a = 2 + 3t$, $b = 4 - t$. Substituting these expressions for a and b into the first of these three equations yields $3(1 + 3t) - 1(2 - t) = 0$, from which $t = -.1$. Therefore $P = (1.7, 4.1)$ and L has the equations: $x = 1.7 + .7s$, $y = 4.1 + 2.1s$.

Two lines: 3-space. The development here is a straightforward generalization from that given for 2-space. As before, the significant formula comes from the consideration of $\triangle OP_1P_2$, where L_1 and L_2 either contain or are parallel to $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$. The results are indicated below, but the proofs, which are not at all difficult, are left to the student.

$$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2$$

$$(3) \quad \text{or} \qquad \cos\theta = \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\,\sqrt{\ell_2^2 + m_2^2 + n_2^2}}$$

As before, the test for perpendicularity becomes

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0 \text{ , or } \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0 .$$

These may be represented simply, in vector form, as

$$[\lambda_1, \mu_1, \nu_1] \cdot [\lambda_2, \mu_2, \nu_2] = 0 \text{ , or } [\ell_1, m_1, n_1] \cdot [\ell_2, m_2, n_2] = 0 .$$

Example 1. Find the angle between two lines having direction cosines as follows:

$$\lambda_1 = \frac{-2}{\sqrt{5}}, \ \mu_1 = 0, \ \nu_1 = \frac{1}{\sqrt{5}}, \text{ and } \lambda_2 = \frac{1}{\sqrt{3}}, \ \mu_2 = \frac{1}{\sqrt{3}}, \ \nu_2 = \frac{1}{\sqrt{3}} .$$

Solution.

$$\cos\theta = \left[-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right] \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

$$= -\frac{1}{\sqrt{15}} \approx -.258$$

$$\therefore \quad \theta \approx 105^\circ$$

Example 2. Show that the lines $L_1 : x = 2 + 3t$, $y = 3 - t$, $z = 2 + 4t$, $L_2 : x = 5 + t$, $y = 6 + 7t$, $z = 7 + t$, are perpendicular to each other.

Solution. $[3,-1,4] \cdot [1,7,1] = (3)(1) + (-1)(7) + (4)(1) = 0$.

Example 3. Find the line L_3 which contains $P = (7,4,5)$ and is perpendicular to L_1 of the previous exercise.

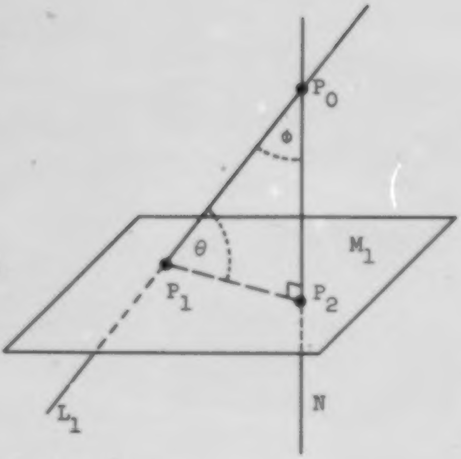
Solution. If L_3 meets L_1 at $P = (a,b,c)$ then we may take, as direction numbers for L_3 , $(a - 7 , b - 4 , c - 5)$. The condition for perpendicularity requires $3(a - 7) - 1(b - 4) + 4(c - 5) = 0$. Since $P = (a,b,c)$ is on L_1 , we have $a = 2 + 3t$, $b = 3 - t$, and $c = 2 + 4t$. If we substitute these expressions for the coordinates into the previous equation we get:

$$3(-5 + 3t) - 1(-1 - t) + 4(-3 + 4t) = 0 \text{ , from which } t = 1 \text{ .}$$

Therefore $P = (5,2,6)$ and L_3 has the equations: $x = 7 + 2t$, $y = 4 + 2t$, $z = 5 - t$.

Line and Plane: L_1 , M_1 . It is convenient to consider the line $L_1 : x = a_1 + \ell_1 t$, $y = b_1 + m_1 t$, $z = c_1 + n_1 t$; and the plane $M_1 : px + qy + rz + s = 0$. We have already developed criteria for L_1 to be parallel, or perpendicular to M_1 . Suppose it is neither, and intersects M_1 at point P_1 . Then any other point of L_1 , say P_0 determines, with M_1 , a unique line N , perpendicular to M_1 , and meeting it at, say, P_2 . We define the angle between L_1 and M_1 to be the angle $P_0P_1P_2$, designated as θ . Note that this definition requires $0^\circ < \theta < 90^\circ$

Since N has direction numbers (p,q,r) and L_1 has direction numbers (ℓ_1,m_1,n_1), we can find the angles between L_1 and N, from Equation (3) of the previous section. We need the acute angle, designated ϕ, and therefore use the absolute value of the right member as $\cos\phi$. But, from right $\Delta P_0P_1P_2$, since θ and ϕ are complementary, we have $\sin\theta = \cos\phi$, and the equation we want:



$$\sin\theta = \frac{|\ell_1 p + m_1 q + n_1 r|}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{p^2 + q^2 + r^2}} \tag{4}$$

Example. Find the angle between $L_1 : x = 2 + t,\ y = 3 - 2t,\ z = 1 + t$; and $M_1 : 3x + 4y - 12z + 5 = 0$.

Solution.

$$\sin\theta = \frac{|1(3) - 2(4) + 1(-12)|}{\sqrt{1^2 + (-2)^2 + 1^2}\ \sqrt{3^2 + 4^2 + (-12)^2}} = \frac{|-17|}{\sqrt{6}\ \sqrt{169}};$$

$$\sin\theta = \frac{17}{13\sqrt{6}} \approx .53 \qquad \therefore \quad \theta \approx 32^\circ$$

Two planes: M_1, M_2. Consider the planes, $M_1 : p_1x + q_1y + r_1z + s_1 = 0$, $M_2 : p_2x + q_2y + r_2z + s_2 = 0$, and a point $P_0 = (a_0,b_0,c_0)$ not lying in either plane. P_0 and M_1 determine a unique normal line N_1, and P_0 and M_2 a unique normal line N_2. We define the angles between planes M_1 and M_2 to be the angles between lines N_1 and N_2. If N_1 and N_2 coincide, then the planes are perpendicular to a common line and must be parallel or coincident. The analytic conditions are easy to find. Since N_1 and N_2 contain a common point P_0, and have direction numbers (p_1,q_1,r_1) and (p_2,q_2,r_2) they will coincide if and only if these direction numbers are equivalent, that is, if there is a number $k \neq 0$, such that $p_1 = kp_2$, $q_1 = kq_2$,

$r_1 = kr_2$; and these are the conditions that M_1 be parallel to or coincident with M_2 , as has been noted earlier. Of course M_1 and M_2 will coincide if and only if, further, $s_1 = ks_2$, otherwise M_1 and M_2 are parallel.

If N_1 and N_2 do not coincide, the angles between them can be found from Equation (3) of the previous section, and these are precisely the angles between M_1 and M_2 :

$$\text{(5)} \qquad \operatorname{Cos} \theta = \frac{p_1p_2 + q_1q_2 + r_1r_2}{\sqrt{p_1^2 + q_1^2 + r_1^2}\ \sqrt{p_2^2 + q_2^2 + r_2^2}}$$

If one of these angles is designated as θ , another must be the supplement of θ , and the remaining two angles congruent to these. Then the right member of Equation (5) gives the cosine either of θ or of its supplement. We are usually interested in the acute angle, in which case we use the absolute value of the right member of (5) .

Example. Find the angles between the planes $M_1 : x - 2y + z - 4 = 0$, and $M_2 : 2x + 2y - z + 3 = 0$.

Solution.

$$\operatorname{Cos} \theta = \frac{1(2) - 2(2) + 1(-1)}{\sqrt{1^2 + (-2)^2 + 1^2}\ \sqrt{2^2 + 2^2 + (-1)^2}}$$

$$= \frac{-3}{\sqrt{6}\ \sqrt{9}} \approx -.41$$

$$\theta \approx 156^\circ$$

$\therefore$ The angles are 156° and 24° .

Example. Find an equation of the plane, perpendicular to line $L : x = 2 + t$, $y = 3 - 2t$, $z = 1 + 3t$, and containing the point $A = (3,1,2)$.

Solution. If $P = (x,y,z)$ is any point of the plane, then direction numbers for $\overleftrightarrow{PA}$ are $(x - 3 , y - 1 , z - 2)$. The condition of perpendicularity requires that

$$1(x - 3) - 2(y - 1) + 3(z - 2) = 0 ,$$

and this is the solution, which may be written more compactly as

$$x - 2y + 3z - 7 = 0 .$$

Exercises D-5

Consider these three lines for Exercises 1 to 4.

$$L_1 : x = 3 - t \ , \ y = 2 + 3t$$

$$L_2 : x = 2 + t \ , \ y = 1 - 2t$$

$$L_3 : x = 1 + 3t \ , \ y = 3 + 2t \ .$$

1. (a) Find the angle between L_1 and L_2.

   (b) Find the angle between L_1 and L_3.

   (c) Find the angle between L_2 and L_3.

2. Find the line through the point $(3,5)$ and perpendicular to

   (a) L_1 (b) L_2 (c) L_3

3. Find the bisectors of the angles formed by L_1 and L_2, using the locus definition of an angle bisector, (points equidistant from the given lines); then show, by the methods of this section, that the angles have been cut into congruent pairs.

4. If L_1, L_2 meet at P_3; L_2, L_3 meet at P_1; and L_3, L_1 at P_2,

   (a) find the coordinates of P_1, P_2, P_3.

   (b) Use these results to find the lines which contain the three altitudes of $\Delta P_1P_2P_3$.

5. At what angles does the line determined by $(1,3)$, $(4,-2)$, meet the line determined by $(-1,2)$, $(2,-3)$?

Consider these lines for Exercises 6 to 14.

$$L_1 : x = 2 - 3t \ , \ y = 3 + t \ , \ z = 4 + 2t$$

$$L_2 : x = 3 + t \ , \ y = 4 - t \ , \ z = 2 + 3t$$

$$L_3 : x = 1 + 2t \ , \ y = 2 + t \ , \ z = 4 - 3t$$

6. Find the angles

   (a) between L_1 and L_2.

   (b) between L_1 and L_3.

   (c) between L_2 and L_3.

7. Find the equations of a line through $P = (1,2,3)$ and perpendicular to

(a) L_1 . (b) L_2 . (c) L_3 .

8. Find equations of a line

(a) N_1 perpendicular to both L_2 and L_3 .

(b) N_2 perpendicular to both L_1 and L_3 .

(c) N_3 perpendicular to both L_1 and L_2 .

9. Find an equation of a plane which contains the point $P = (3,5,7)$ and is perpendicular to

(a) L_1 . (b) L_2 . (c) L_3 .

10. Find an equation of a plane which

(a) contains L_1 and is parallel to L_2 .

(b) contains L_1 and is parallel to L_3 .

(c) contains L_2 and is parallel to L_1 .

(d) contains L_2 and is parallel to L_3 .

(e) contains L_3 and is parallel to L_1 .

(f) contains L_3 and is parallel to L_2 .

Consider these planes

$$M_1 : 2x + 3y - z + 5 = 0$$

$$M_2 : 3x - y + 2z - 4 = 0$$

$$M_3 : x + 2y + 3z + 7 = 0$$

11. Find the angles between

(a) M_1 , M_2 (b) M_1 , M_3 (c) M_2 , M_3

12. Find the plane which

(a) contains L_1 and is perpendicular to M_1 .

(b) contains L_1 and is perpendicular to M_2 .

(c) contains L_1 and is perpendicular to M_3 .

(d) contains L_2 and is perpendicular to M_1 .

(e) contains L_2 and is perpendicular to M_2 .

(f) contains L_2 and is perpendicular to M_3 .

(g) contains L_3 and is perpendicular to M_1 .

(h) contains L_3 and is perpendicular to M_2 .

(i) contains L_3 and is perpendicular to M_3 .

13. Find the plane which contains the origin, and is perpendicular to the line determined by

(a) M_1 , M_2

(b) M_1 , M_3

(c) M_2 , M_3

14. Find the angles between each of the lines L_1 , L_2 , L_3 , given above, and each of the planes, M_1 , M_2 , M_3 :

(a) L_1M_1

(b) L_1M_2

(c) L_1M_3

(d) L_2M_1

(e) L_2M_2

(f) L_2M_3

(g) L_3M_1

(h) L_3M_2

(i) L_3M_3

15. Find the angle that each axis makes with each plane.

(a) M_1

(b) M_2

(c) M_3

16. Consider two intersecting lines in 2-space, whose equations are

$L_1 : a_1x + b_1y + c_1 = f_1(x,y) = 0$, and

$L_2 : a_2x + b_2y + c_2 = f_2(x,y) = 0$. Develop a formula for the cosine of one of the angles between them, in terms of a_1 , b_1 , c_1 , a_2 , b_2 , c_2 .

Supplement to Chapter 7

Part 1

CONIC SECTIONS

S7-1. Cones and Sections of Cones

In your study of geometry you learned that a circular cone may be defined as the union of all segments $\overline{VP}$ where P is any point contained in a circular region C and V is any point of space not contained in the plane of C. The resulting geometric configuration is a solid. If O is the center of C and if $\overleftrightarrow{OV}$ is perpendicular to the plane of C, the resulting solid is a right circular cone.

An alternative idea of a cone is as an unbounded surface rather than as a bounded solid.

DEFINITIONS. Let D be a curve contained in a plane E and let V be any point not in E. Then the union of all lines $\overleftrightarrow{VP}$ where P is a point of D, is a cone.

The curve D is a plane curve and the directrix of the cone; the point V is the vertex of the cone; the lines $\overleftrightarrow{VP}$ are the elements of the cone. Note that according to this definition of a cone the surface falls naturally into two parts.

DEFINITION. If V is the vertex of a cone, D is the directrix of the cone, and P is any point of D, then the union of the rays $\overleftrightarrow{VP}$ is a nappe of the cone; the union of the rays opposite to $\overleftrightarrow{VP}$, is also a nappe of the cone.

It becomes apparent that while a given cone has a unique vertex, it has infinitely many possible directrices.

Cones may be named after curves which are their directrices. Thus a cone which has a circle as a directrix is called a circular cone. The line containing the vertex of the cone and the center of the circle is called the axis of the cone. If the axis of the cone is perpendicular to the plane of the circle, then the cone is called a right circular cone. The right circular cones are the cones which we shall consider. We state two theorems with the proofs suggested as exercises.

THEOREM S7-1. A circular cone is a right circular cone if and only if the points of a directrix are equidistant from the vertex.

THEOREM S7-2. The points of the axis of a right circular cone are equidistant from the elements of the cone.

The intersection of a surface and a plane is called a section of the surface. If the surface is directed or generated by a plane curve (as are cones, prisms, cylinders, and pyramids), then the sections of the surface formed by planes parallel to the plane of the generating curve are called cross-sections of the surface. If the surface has an axis, then the sections of the surface formed by planes perpendicular to the axis are called right-sections. Since the axis of a right circular cone is perpendicular to the plane of the directrix, the cross-sections and right-sections are identical. The sections of a right circular cone are called conic sections. They may also be obtained from other cones and surfaces. This will be made clear in Chapter 9. However, we shall confine our approach here to sections of right circular cones.

What we plan to do is to use geometric methods to discover certain characteristics of the conic sections. These characteristics enable us to use analytic methods to study the conic sections as curves in the intersecting plane.

Exercises S7-1

1. Prove Theorem S7-1.
2. Prove Theorem S7-2.

87-2. Tangent Spheres and Cutting Planes

Let us consider the sections of a right circular cone. For the time being we shall not consider those sections which contain the vertex of the cone. Such sections are classified as *degenerate conic sections* and will be studied separately. Let V be the vertex of the cone, $\underline{a}$ the axis of the cone, and E the intersecting or *cutting plane*. There are associated with each section one or more spheres with center on the axis $\underline{a}$ which are tangent both to E and to all the elements of the cone. It is our first task to prove the existence of such a sphere or spheres.

From the definition of a right circular cone, it follows that any two elements of the cone form congruent acute angles with the axis. We define the measure of these acute angles to be the *elemental angle* of the cone, which we denote by x.

We recall that the distance from a point to a line is the length of a segment which is perpendicular to the line and of which the end points are the given point and a point in the line. Also, the distance from a point to a plane is the length of a segment which is perpendicular to the plane and of which the end points are the given point and a point in the plane.

The axis of the cone is the set of all points which are equidistant from the elements of the cone. We say therefore that each point of the axis is the same distance from the cone and that this distance is the distance between the point and the cone.

Given any real number except zero, there exist two points on the axis which are this measure of distance from the cone, one on either side of the vertex. For the real number zero there exists only one such point, the vertex of the cone. For each of these points on the axis, the points of the cone at the given distance lie in the same plane and form a circle. Since these are the closest points of the cone, there is a sphere with center at the given point and radius equal to the given distance, which is tangent to each element of the cone. For this reason we say that the sphere is *tangent to the cone*. The union of the points of tangency is a circle, called the *circle of tangency*.

We turn our attention to the plane intersecting the cone. This plane may be parallel to the axis of the cone, but in all other cases it intersects the axis, either in the axis itself or in a set containing a single point. We first consider intersections in a single point.

If the cutting plane is not perpendicular to the axis of the cone, then a pair of congruent acute vertical angles is formed by the axis of the cone and its projection in the cutting plane. We define the measure of these acute angles to be the cutting angle of the plane. If the cutting plane is perpendicular to the axis of the cone, we define the cutting angle to be $\frac{\pi}{2}$ in radian measure or 90 in degree measure. If the cutting plane is parallel to the axis of the cone (in this case it may contain the axis), then the cutting angle is defined to be zero. (We could avoid defining these angles in such an unnatural way, were we to consider parallel planes containing the vertex of the cone. However, we are interested solely in the measures of these angles and adopt these definitions.)

Exercises S7-2

1. Prove that any two elements of a right circular cone form congruent acute angles with the axis of the cone.

* 2. Prove that the axis of a right circular cone is the locus of points equidistant from the elements of the cone.

3. Prove that, given any real number except zero as a measure of distance, there exist two distinct points on the axis of a right circular cone which are this measure of distance from the cone.

4. Prove that if a point P on the axis of a right circular cone is at a distance d from the cone, then the locus of points of the cone at a distance d from P is a circle.

S7-3. Spheres of Tangency

Figure 1 is a schematic representation of a plane cutting a cone from a point of view parallel to the cutting plane. V is the vertex of the cone, a is the axis of the cone, ℓ and ℓ' are elements of the cone, α is the elemental angle, β is the cutting angle, P is the point of intersection of the cutting plane and the axis of the cone, and m is the projection of the axis in the cutting plane.

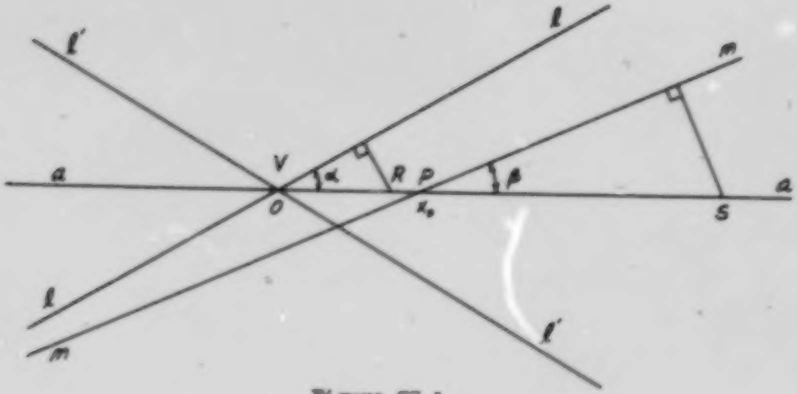


Figure S7-1

We consider three different coordinate systems on line $\underline{a}$. In the first coordinate system X the origin is at V ; the coordinate of P is positive and is denoted by x_0 . The coordinate of an arbitrary point is denoted by x .

The second coordinate system x' is oriented from V to P and assigns to each point R as its coordinate x' the distance from R to the cone, and consequently the radius of the sphere tangent to the cone with center R . This is the case to the right of V . The origin is at V . To the left of V the coordinate is the negative of this radius. This coordinate system is related to the first coordinate system by the following linear equation:

$$x' = x \sin\alpha \, .$$

The third coordinate system x" on $\underline{a}$ is oriented from P to V and assigns to each point S as its coordinate x" the distance from S to the cutting plane, and consequently the radius of a sphere tangent to the cutting plane with center S . This will be the case to the left of P . The origin is at P . To the right of P the coordinate is the negative of this radius. This coordinate system is related to the first coordinate system by the following linear equation:

$$(x_0 - x)x'' = \sin\beta$$

We observe that, if x' = x" , the corresponding point on $\underline{a}$ is the center of the sphere tangent to the cone and the cutting plane. This is the desired sphere mentioned in Section S7-2.

We equate these two expressions and solve for x :

$$(x_0 - x)x \sin\alpha = \sin\beta$$

$$= x_0 \sin\beta - x \sin\beta$$

$$x \sin\alpha + x \sin\beta = x_0 \sin\beta$$

$$x = x_0 \frac{\sin\beta}{\sin\alpha + \sin\beta}$$

We note that we oriented the first coordinate system in such a way that x_0 was positive and that, inasmuch as α and β are measures of acute angles, $\frac{\sin\beta}{(\sin\alpha + \sin\beta)}$ is between 0 and 1 . Hence x is the coordinate of a point between V and P and the radius of the sphere is $x_0\left(\frac{\sin\alpha \sin\beta}{\sin\alpha + \sin\beta}\right)$.

If $\beta > \alpha$, then $\sin\beta > \sin\alpha$, and we discover a second sphere tangent both to the cone and to the cutting plane, but with its center to the right of P . To the right of P the radius of a sphere tangent to the plane is $-x''$. If $x' = -x''$,

$$x \sin\alpha = -(x_0 - x)\sin\beta$$

and

$$x = x_0\left(\frac{\sin\beta}{\sin\alpha - \sin\beta}\right) .$$

Since $\frac{\sin\beta}{(\sin\beta - \sin\alpha)} > 1$, x is the coordinate of a point to the right of P . The radius of the sphere is $x_0\left(\frac{\sin\alpha \sin\beta}{\sin\beta - \sin\alpha}\right)$.

If $\beta < \alpha$, then $\sin\beta < \sin\alpha$; we discover a second sphere with center to the left of V . To the left of V the radius of a sphere tangent to the cone is $-x'$. If $-x' = x''$,

$$-x \sin\alpha = (x_0 - x)\sin\beta$$

and

$$x = -x_0\left(\frac{\sin\beta}{\sin\alpha - \sin\beta}\right)$$

where $\frac{\sin\beta}{(\sin\alpha - \sin\beta)} > 1$. Thus x is the coordinate of a point to the left of V , the center of the sphere is more remote from the origin than was that of the first sphere, and the radius is $x_0\left(\frac{\sin\alpha \sin\beta}{\sin\alpha - \sin\beta}\right)$.

If $\beta = \alpha$, $\sin\beta = \sin\alpha$, and the search for other spheres is in vain. The coefficients of x_0 are not defined outside the segment $\overline{VP}$.

Lastly, we consider the possibility that the cutting plane may be parallel to the axis of the cone. In this case the distance from a point on the axis to the plane is constant. Thus $x'' = k$, and following the above argument, we discover that $x = \pm \frac{k}{\sin\alpha}$; there are two spheres, one on either side of V, and each with radius k. We recall that the cutting angle is zero in this case, for the cutting angle is not really the angle itself, but rather a measure associated with the angle.

S7-4. Degenerate Conic Sections

Before continuing with our discussion of the more elaborate conic sections, we may digress to consider what happens if the cutting plane contains the vertex of the cone. A geometric description should be sufficient. If $\beta > \alpha$, then the vertex is the only point of the section. If $\beta = \alpha$, then the section is a single element of the cone, that is, a line. If $\beta < \alpha$, the section is the union of two elements of the cone, that is, the union of two intersecting lines.

Some sections of the surface called a right circular cylinder are sections of right circular cones. The exceptions are those sections obtained by a cutting plane parallel to the axis of the cylinder, with distance from the axis less than the radius of the cylinder. (The plane may contain the axis.) These sections are the union of two parallel lines. Though not obtainable as sections of cones for algebraic reasons they are included among the degenerate conic sections.

S7-5. Geometric Properties of the Conic Sections

From our consideration of the conic sections so far we may make certain general observations. If $\beta = \frac{\pi}{2}$ (in radians) or 90 (in degrees), it is intuitively obvious and not difficult to prove that this section is a circle. If $\frac{\pi}{2} > \beta > \alpha$, it is apparent that the plane cuts every element of one nappe and that the resulting section is a closed curve. If $\beta = \alpha$, the plane cuts some, but not all of the elements of one nappe. Lastly, if $\beta < \alpha$, the plane cuts some, but not all the elements of each nappe and the curve has two distinct branches.

But to continue our study we need more information. We consider Figure S7-2. We are given a right circular cone with vertex V, axis a, and elemental angle α. E is a cutting plane, not containing V, with an acute cutting angle β. The conic section is the curve s. The tangent sphere with center O is tangent to the cone in circle c and to the cutting plane at point F.

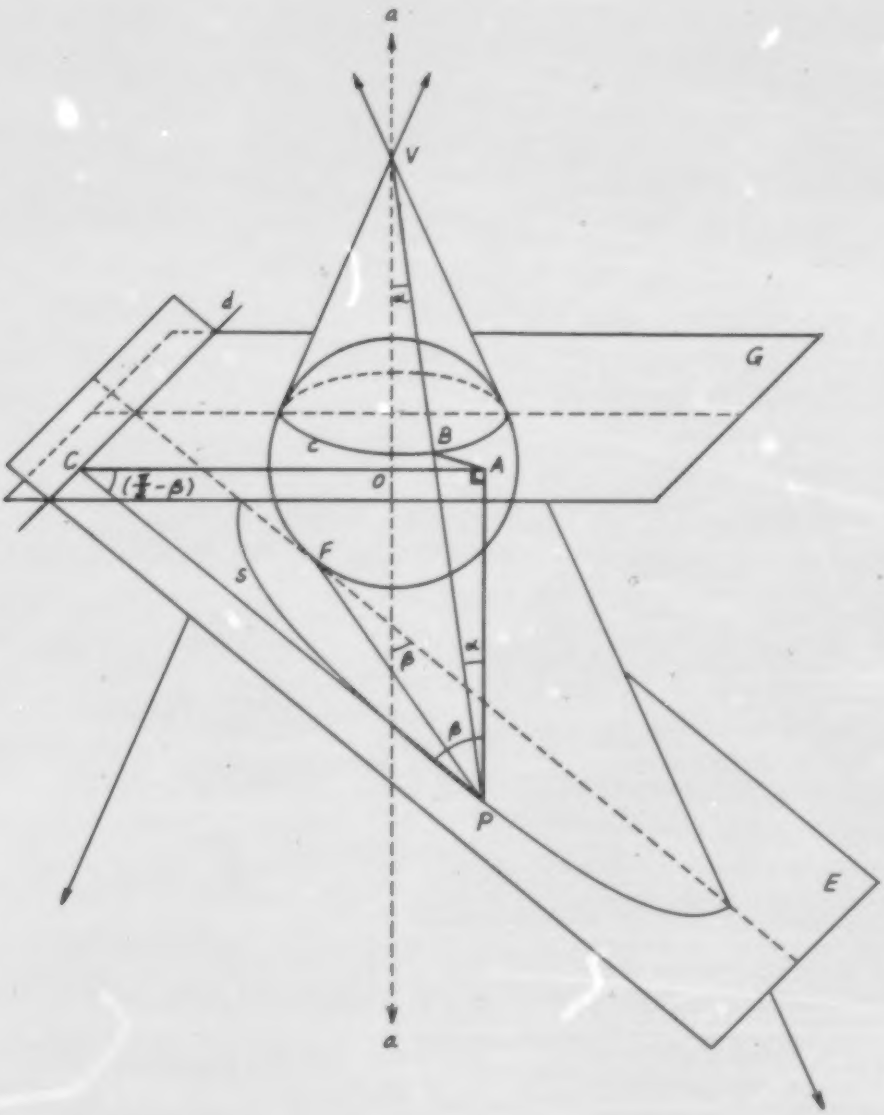


Figure S7-2

Let G be the plane containing circle c . G is perpendicular to the axis a , and since E is assumed not to be perpendicular to a , G and E must form a dihedral angle with edge d . The plane angle of the dihedral angle is complementary to the cutting angle and has measure $(\frac{\pi}{2} - \beta)$.

Let P be any point of the conic section s . The plane containing P and perpendicular to d intersects the dihedral angle in a plane angle of the dihedral angle which has vertex C and measure $(\frac{\pi}{2} - \beta)$. Let A be the foot of the perpendicular from P to the other side of the plane angle. $\overleftrightarrow{PA}$ is perpendicular to G and $\triangle PAC$ is a right triangle. Since $m \angle PCA = (\frac{\pi}{2} - \beta)$, $m \angle APC = \beta$ and

$$\cos\beta = \frac{d(A,P)}{d(P,C)} . \tag{1}$$

We observed that $\overleftrightarrow{AP}$ was perpendicular to G . The axis a is also perpendicular to G , so a and $\overleftrightarrow{AP}$ are parallel. Consider the element of the cone $\overleftrightarrow{PV}$ which intersects the circle of tangency c in point B (which is in G) . Since the tangent sphere is between V and the cutting plane, B is between V and P . The elemental angle and $\angle APB$ are a pair of alternate interior angles formed by a transversal of two parallel lines, and consequently $m \angle APB = \alpha$. $\triangle APB$ is a right triangle and

$$\cos\alpha = \frac{d(A,P)}{d(P,B)} . \tag{2}$$

Both $\overline{PB}$ and $\overline{PF}$ are tangent segments to the sphere from the same point and hence $d(P,F) = d(P,B)$. Substituting in (2) , we obtain

$$\cos\alpha = \frac{d(A,P)}{d(P,F)} . \tag{3}$$

Dividing (1) by (3) , we obtain

$$\frac{\cos\beta}{\cos\alpha} = \frac{d(P,F)}{d(P,C)} . \tag{4}$$

Since both β and α are constant for a given conic section, this quotient is a constant. It is called the eccentricity of the conic section and is denoted by the small letter e . Geometrically this means that for any point of a given conic section the ratio of its distance from a well-defined point to its distance from a well-defined line is a constant. Both the point, which is called the focus or focal point, and the line, which is called the directrix, lie in the plane of the conic section. Since we have taken both the elemental angle and the cutting angle to be the measures of acute angles, the eccentricity e will be a positive real number.

We have observed that it is perfectly possible for the cutting plane E to be perpendicular to the axis of the cone. In this case E and G are parallel and the section has no directrix. It does have a focus which is the intersection of the cutting plane and the axis. The section is a circle and the center is at the focus; if U is the focus, then the radius of the circle is $d(U,V) \cdot \tan\alpha$. In this case the expression for the eccentricity would be

$$\frac{\cos(\frac{\pi}{2})}{\cos\alpha}$$, which is zero.

Since this is distinct from the other cases, we may accept it without inconsistency.

We observe that if $\frac{\pi}{2} > \beta > \alpha$, $\cos\beta < \cos\alpha$ and $e < 1$; if $\beta = \alpha$, $\cos\beta = \cos\alpha$ and $e = 1$; if $0 \leq \beta < \alpha$, $1 \geq \cos\beta > \cos\alpha$ and $e > 1$. We take these properties to be definitive for the conic sections.

DEFINITIONS. Given a conic section with eccentricity e:
The conic section is an ellipse if $0 < e < 1$.
The conic section is a parabola if $e = 1$.
The conic section is a hyperbola if $e > 1$.
The conic section is a circle if $e = 0$.

On the other hand, we have shown they may be described by their geometric properties. A circle is the locus of points in a plane at a given distance from a given point, called the center; an ellipse is the locus of points in a plane such that for each point the ratio of its distance from a given point to its distance from a given line is a constant which is less than one; a parabola is the locus of points in a plane such that for each point the ratio of its distance from a given point to its distance from a given line is one; a hyperbola is the locus of points in a plane such that for each point the ratio of its distance from a given point to its distance from a given line is a constant which is greater than one.

Exercises S7-5

1. Prove that if a cutting plane is perpendicular to the axis of a right circular cone, then the sphere of tangency is tangent to the plane at a point on the axis. Prove that in this case the conic section is a circle which centers on the axis.

*2. In Section S7-3 we discovered that if $\beta > \alpha$, there exists a second sphere of tangency such that its center is on the other side of the cutting plane from the vertex. Let this sphere be tangent to the cutting plane at F'. Prove that if P is a point of the section, then $d(P,F) + d(P,F')$ is a fixed constant. In other words, prove that an ellipse is the locus of points in a plane such that for each point, the sum of its distances from two given points in the plane is a fixed constant. (Hint: In Figure S7-2 the second sphere lies below the cutting plane; let c' be its circle of tangency. Let B' be the intersection of $\overleftrightarrow{VP}$ and c'. Then prove that $d(P,F) + d(P,F') = d(B,B')$. Then prove that this distance is the same for all P.)

3. In Section S7-3 we discovered that if $\beta < \alpha$, there exists a second sphere of tangency such that the vertex lies between the centers of the two spheres. Let this sphere be tangent to the cutting plane at F'. Prove that if P is a point of the section, then $|d(P,F) - d(P,F')|$ is a fixed constant. In other words prove that a hyperbola is the locus of points in a plane such that for each point, the absolute value of the difference between its distances from two given points in the plane is a fixed constant. (Hint: In Figure S7-2, the second sphere lies within the upper nappe of the cone; let c' be its circle of tangency. Let B' be the intersection of $\overleftrightarrow{VP}$ and c'. Then prove that $|d(P,F) - d(P,F')| = d(B,B')$. Then prove that this distance is the same for all P.)

*4. Let C be a circle contained in a plane E. The union of the lines perpendicular to E which contain points of C is a right circular cylinder. The lines are called elements of the cylinder; the circle is called a directrix of the cylinder. Prove that the sections of a right circular cylinder are conic sections. Show that in the case of the right circular cylinder there are also spheres of tangency (i.e. tangent to the cylinder in a circle and to the cutting plane at a focal point of the conic section).

In general, the sections of any cone or cylinder, with a conic section as directrix, are also conic sections.

Part 2

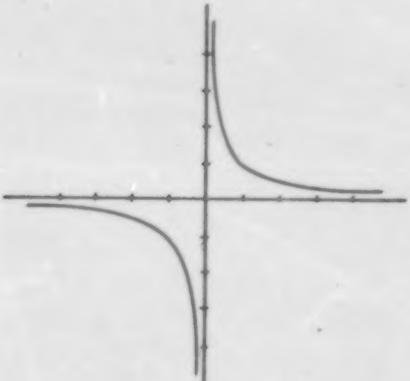
THE GENERAL SECOND-DEGREE EQUATION

S7-6. The General Second-Degree Equation; Rotations and Translations

The conic sections which we have studied have been represented in rectangular coordinates by second-degree equations in two variables. It seems natural to ask whether all equations of second degree in x and y have loci which are conic sections. In its most general form such an equation may be written as

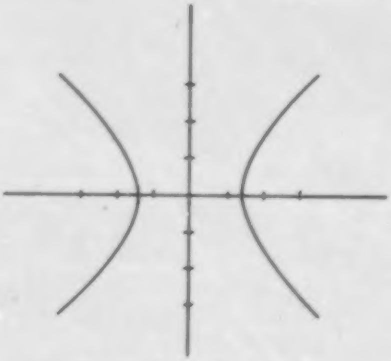
(1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A , B , and C are not all zero.

This general form may be difficult to identify, but some techniques which we have used in the preceding sections will permit us to simplify it. The major stumbling block is posed by the xy-term. The only previous equation containing an xy-term, which we have considered in detail, was that of an equilateral hyperbola. We also have another equation for an equilateral hyperbola. Let us consider the graphs of $xy = 1$ and of $\frac{x^2}{2} - \frac{y^2}{2} = 1$.



$xy = 1$

Figure S7-6a



$\frac{x^2}{2} - \frac{y^2}{2} = 1$

Figure S7-6b

The graphs of these two equations seem remarkably similar. Not only are the asymptotes perpendicular in each case, but also the transverse axes are congruent. In fact, it would appear that the graph in Figure S7-6b may be obtained from that in Figure S7-6a by a clockwise rotation of axes through an angle of 45°. The first equation contains an xy-term, while the second does not. The suggestion is that a rotation of axes as described in Section 4-8 might result in the elimination of the xy-term. It turns out that this is the case, but we are now faced with a second question. What size rotation should we consider? Let us consider the effect of any rotation of axes on the general second-degree equation. We recall that the equations of rotation are:

$$x = x' \cos \theta - y' \sin \theta$$
$$y = x' \sin \theta + y' \cos \theta .$$

If we substitute these values in Equation (1) and expand, we obtain

$$A(x'^2 \cos^2 \theta - 2x'y' \sin \theta \cos \theta + y'^2 \sin^2 \theta)$$
$$+ B(x'^2 \sin \theta \cos \theta - x'y' \sin^2 \theta + x'y' \cos^2 \theta - y'^2 \sin \theta \cos \theta)$$
$$+ C(x'^2 \sin^2 \theta + 2x'y' \sin \theta \cos \theta + y'^2 \cos^2 \theta) + D(x' \cos \theta - y' \sin \theta)$$
$$+ E(x' \sin \theta + y' \cos \theta) + F = 0$$

However, all we want to know is the coefficient of the $x'y'$-term. This is

$$-2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta .$$

If this coefficient is zero, the transformed equation will not contain any $x'y'$-term. If

$$-2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta = 0 ,$$

then

$$B(\cos^2 \theta - \sin^2 \theta) = 2(A - C) \sin \theta \cos \theta .$$

We recall that $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$ and that $2 \sin \theta \cos \theta = \sin 2\theta$. Thus we may write

$$B \cos 2\theta = (A - C) \sin 2\theta ,$$

or, if $A \neq C$,

$$\frac{B}{A - C} = \frac{\sin 2\theta}{\cos 2\theta}$$

or

$$\frac{B}{A - C} = \tan 2\theta .$$

If $A = C$, then

$$B \cos 2\theta = 0$$

or

$$\cos 2\theta = 0 .$$

(We recall that if B were zero, we would not have had to go to all this trouble.) In either case, all we require is a single value of θ which satisfies the appropriate condition. If $\cos 2\theta = 0$, 2θ may be 90°; thus θ may be 45°. If $\tan 2\theta = \frac{B}{A - C}$, which is not zero, we recall that the tangent assumes all non-zero real values once and only once between 0° and 180°. Thus, there exists a unique acute angle θ such that $\tan 2\theta = \frac{B}{A - C}$.

Thus we have shown that in every case in which the second-degree equation has an xy-term, it can be transformed, by a rotation of axes through a unique acute angle, to an equation without an xy-term. The transformed equation has the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$, or, dropping the primes, the form

$$(2) \quad Ax^2 + Cy^2 + Dx + Ey + F = 0 . \quad (A \text{ and } C \text{ are never both zero.})$$

Now the equation is in a form which may be identified more easily. We have already developed techniques for simplifying equations of this form. It is proper to drop the primes only when the form of the equation is being studied.

If AC is not zero, we first complete the squares for the x^2- and x-terms and y^2- and y-terms to obtain

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = \frac{D^2}{4A} + \frac{E^2}{4C} - F, \ AC \neq 0$$

or

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}, \ AC \neq 0 .$$

Now a translation of axes, as introduced in Section 10-2 and described by the equations

$$x = x' - \frac{D}{2A}$$

$$y = y' - \frac{E}{2C},$$

gives the transformed equation

$$Ax'^2 + Cy'^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}, \ AC \neq 0,$$

in which the primes have been omitted for simplicity.

We recognize that if AC is negative and $\frac{CD^2 + AE^2 - 4ACF}{4AC}$ is not zero, or if A, C, and $\frac{CD^2 + AE^2 - 4ACF}{4AC}$ are all positive or all negative, the transformed equation is the equation of a conic section. If A equals C, the conic section is a circle; if AC is positive and A is not equal to C, the conic section is an ellipse; if AC is negative, the conic section is an hyperbola.

We must also consider the case in which $AC = 0$ in Equation (2). Suppose A is zero. Then C is not zero, and we may complete the square for the y^2- and y-terms. Equation (2) is now

$$Cy^2 + Dx + Ey + F = 0,$$

which becomes

$$Cy^2 + Ey = -Dx - F$$

or

$$C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -D\left(x + \frac{F}{D} - \frac{E^2}{4CD}\right)$$

or

$$\left(y + \frac{E}{2C}\right)^2 = -\frac{D}{C}\left(x - \frac{E^2 - 4CF}{4CD}\right).$$

A translation of axes, described by the equations

$$x = x' + \frac{E^2 - 4CF}{4CD}$$

$$y = y' - \frac{E}{2C},$$

gives the transformed equation

$$y^2 = -\frac{D}{C}x .$$

We recognize this as the equation of a parabola, with the vertex at the origin and the axis on the x-axis.

If C is zero, a similar development may be made. The resulting equation will again be of a parabola with the vertex at the origin, but the axis will be on the y-axis.

Exercises 7-6

1. Through what angle must the axes be rotated to eliminate the xy-term from each of the following equations ?

   (a) $x^2 - 4xy + 4y^2 - 4x - 7 = 0$

   (b) $x^2 + \sqrt{3}\, xy + 2y^2 - 3 = 0$

   (c) $x^2 - 3xy + 4y^2 - 9 = 0$

   (d) $x^2 + 3xy - x + y - 1 = 0$

   (e) $3x^2 + 2\sqrt{3}\, xy + y^2 - 2x - 2\sqrt{3} - 16 = 0$

   (f) $12xy + 9y^2 - 2x - 3y - 10 = 0$

2. For each of the following, simplify the equation, identify the conic section, and draw its graph:

   (a) $5x^2 - 6xy + 5y^2 - 8 = 0$

   (b) $5x^2 - 6xy + 5y^2 + 4x + 4y - 4 = 0$

   (c) $7x^2 + 2\sqrt{3}\, xy + 5y^2 - 16 = 0$

   (d) $3x^2 + 2xy + 3y^2 + 4x + 4y = 0$

   (e) $x^2 - 6xy + y^2 + 14x + 10y + 14 = 0$

   (f) $11x^2 + 24xy + 4y^2 - 44x - 48y + 24 = 0$

   (g) $2xy + 4x - 4y - 9 = 0$

   (h) $9x^2 - 24xy + 16y^2 + 90x + 130y = 0$

This treatment of the quadratic equations which describe conic sections has been solely concerned with techniques employed in simplifying the equations. It is important that we also consider what we have done from a geometric point of view.

In Section 6-2 we have stressed the importance of recognizing symmetries in figures, both as an aid in the sketching of graphs of equations and as a guide in the selection and orientation of a coordinate system to describe a graph by an equation. In particular we have considered axes of symmetry and points of symmetry. We have observed that in rectangular coordinates the y-axis is an axis of symmetry for a locus described by $f(x,y) = 0$ if $f(x,y) = f(-x,y)$ and that the x-axis is an axis of symmetry if $f(x,y) = f(x,-y)$. The origin is a point of symmetry if $f(x,y) = f(-x,-y)$.

The origin is always a point of symmetry if both the x-axis and the y-axis are axes of symmetry. However, the converse of this last statement is not true. (Consider $y = x^3$.)

It was in Section 10-2 that we first overtly considered translations of axes as a means to simplify the analysis of the graph of an equation. However, we have really used this technique before. Do you recall that in Chapter 2 in our discussion of direction angles and direction cosines for a line we found it convenient to consider a parallel line through the origin?

In our rather mechanical treatment of quadratic equations in this section we have been guided by symmetries in the graphs of the equations. The rotations of axes which we performed in Section 10-3 made an axis of symmetry parallel to a coordinate axis. The translations of axes made a point of symmetry also be the origin. (In the case of the parabola there is no point of symmetry. The translation of axes made the vertex be the origin as well.)

It is possible to describe points and axes of symmetry quite generally.

> *DEFINITIONS*. Let S be a set of points. The segments joining points of S are *chords* of the set. If there exists a point P such that, for each point X of S, the segment with end-point X and mid-point P is a chord of the set, then P is a *point of symmetry* or *center* of S.
>
> Let S be a set of points in a plane and let L be a line in the plane. If, for every point X of S, the segment which
>
> (i) has end-point X,
> (ii) is perpendicular to L,
> and (iii) has its mid-point on L,
>
> is a chord of S, then L is an *axis of symmetry* of S.

S7-7. The General Second-Degree Equation, Translation and Rotation

In simplifying second-degree equations, it is in some cases more convenient to translate the axes first to eliminate the x- and y-terms. Then we rotate the new axes to eliminate the xy-term.

If we start again with Equation (1) of Section S7-6 and use the equations of translation

$$x = x' + h$$
$$y = y' + k ,$$

we obtain

$$A(x'^2 + 2hx' + h^2) + B(x'y' + kx' + hy' + hk) + C(y'^2 + 2ky' + k^2)$$
$$+ D(x' + h) + E(y' + k) + F = 0 .$$

If we collect terms, this becomes

$$(1) \quad Ax'^2 + Bx'y' + Cy'^2 + (2Ah + Bk + D)x' + (Bh + 2Ck + E)y'$$
$$+ (Ah^2 + Bhk + Ck^2 + Dh + Ek + F) = 0 .$$

We note that the coefficients of the second-degree terms will not be changed by a translation of axes. If we can find values of h and k such that

$$2Ah + Bk + D = 0$$

and

$$Bh + 2Ck + E = 0 ,$$

we shall be able to substitute these values in Equation (1) to obtain a transformed equation free of first-degree terms. We can solve this pair of equations to obtain

$$h = \frac{\begin{vmatrix} -D & B \\ -E & 2C \end{vmatrix}}{\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}} \qquad \text{and} \qquad k = \frac{\begin{vmatrix} 2A & -D \\ B & -E \end{vmatrix}}{\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix}}$$

if

$$\delta = \begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 4AC - B^2 \neq 0 .$$

The determinant δ is of some interest in the analysis of the second-degree equation and is sometimes called the characteristic.

You should sense that, when it is possible, it is easier to translate the axes first and then perform a rotation of the new axes. The fewer terms there are in an equation, the easier it is to perform a rotation. However, if the characteristic is zero, we cannot find the appropriate values of h and k. We have no choice but to follow the procedure of Section 6-8.

If the characteristic is not zero, the transformed equation is

$$Ax'^2 + Bx'y' + Cy'^2 + F' = 0$$

where

$$F' = Ah^2 + Bhk + Ck^2 + Dh + Ek + F .$$

It is easy to remember what F' is if you notice that when we represent the original equation by $f(x,y) = 0$, then $F' = f(h,k)$.

Exercises S7-7a

1. Find h and k such that a translation of axes described by

$$x' = x + h$$
$$y' = y + k$$

will eliminate the first-degree terms of

$$4x^2 + y^2 - 8x + 4y + 4 = 0 .$$

Verify for this case that the constant term in the transformed equation is equal to $f(h,k)$.

2. Transform each of the following equations by first translating the axes so as to eliminate the first-degree terms. Then rotate the axes to remove the xy-term. Sketch the curve, showing old and new axes.

   (a) $8x^2 - 4xy + 5y^2 - 24x + 24y = 0$

   (b) $3x^2 + 10xy + 3y^2 - 6x + 22y - 53 = 0$

   (c) $7x^2 - 24xy + 120x + 144 = 0$

   (d) $4x^2 - 8xy + 4y^2 - 9\sqrt{2}\,x + 7\sqrt{2}\,y + 14 = 0$

Once again it's important that we consider this method of simplifying the second-degree from a geometric point of view. Why can't we find an appropriate translation of axes when the characteristic is zero? You should recall that in the previous section we observed that the translation of axes makes the new origin a point of symmetry. Our search for values of h and k is in fact a search for the coordinates of a point of symmetry. Since the parabola has no point of symmetry, the characteristic of its equation turns out to be zero. The converse of this statement is not necessarily true, but we shall defer the consideration of this question.

If we approach the analysis of the second-degree equation from a geometric point of view, we can develop methods which may be applied to more complicated problems.

First we observe that if a set of points in a plane has an axis of symmetry, then the axis of symmetry is the perpendicular bisector of chords joining pairs of points of the set. In fact, every point of the set is an endpoint of such a chord. We have already noted that the equation of a locus is frequently simplified if an axis of symmetry of the locus is parallel to one of the coordinate axes. We shall first find an axis of symmetry for the graph of the second-degree equation and then rotate the axes to make one of them parallel to this axis of symmetry. Since the chords in the definition of an axis of symmetry are all perpendicular to the axis of symmetry, they are parallel to each other. Then the lines determined by the chords have parametric representations in terms of a fixed pair of direction cosines (λ,μ). Let (x',y') be the midpoint of a chord. Then the parametric representation of the line containing the chord is

$$\begin{aligned} x &= x' + \lambda t \\ y &= y' + \mu t \ . \end{aligned}$$

When (x,y) is an endpoint of the chord, the coordinates should satisfy the second-degree equation. If we substitute the parametric representation of the endpoint in the second-degree equation, we obtain

$$\begin{aligned} &A(x'^2 + 2\lambda t x' + \lambda^2 t^2) + B(x'y' + \mu t x' + \lambda t y' + \lambda\mu t^2) \\ &+ C(y'^2 + 2\mu t y' + \mu^2 t^2) + D(x' + t) + E(y' + \mu t) + F = 0 \ . \end{aligned}$$

If we collect terms in t^2, λt and μt, we obtain

$$\begin{aligned} (2)\quad &(A\lambda^2 + B\lambda\mu + C\mu^2)t^2 + (2Ax' + By' + D)\lambda t + (Bx' + 2Cy' + E)\mu t \\ &\qquad + (Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F) = 0 \ . \end{aligned}$$

Now we observe that both endpoints of the chord must satisfy the equation. Furthermore, if t_1 is the value of the parameter at one endpoint, $-t_1$ is the value of the parameter at the other endpoint. This must be the case for any chord and any equation. This implies that the form of the equation in t must always be

$$t^2 - t_1^2 = 0 \ .$$

Thus in Equation (2) the coefficient of t, or

$$(3) \qquad (2Ax' + By' + D)\lambda + (Bx' + 2Cy' + E)\mu ,$$

must be zero. Now λ and μ are fixed for any particular second-degree equation, but x' and y' are variables, designating the coordinates of the midpoints of the chords perpendicular to the axis of symmetry. But the midpoints of the chords are on the axis of symmetry. Thus the condition on Expression (3) written as a linear equation in x' and y' is the equation of the axis of symmetry:

$$(4) \qquad (2A\lambda + B\mu)x' + (B\lambda + 2C\mu)y' + (D\lambda + E\mu) = 0 .$$

This equation is in the general form. Hence, $(2A\lambda + B\mu, B\lambda + 2C\mu)$ is a pair of direction numbers for normals to the axis of symmetry. But so is (λ,μ). Therefore, for some non-zero real number k

$$(5) \qquad 2A\lambda + B\mu = k\lambda$$

and

$$B\lambda + 2C\mu = k\mu .$$

If we solve the second equation for μ, we obtain

$$\mu = \frac{-B}{2C - k} \cdot \lambda .$$

We substitute in the first equation, which becomes

$$(2A - k)\lambda - \frac{B^2}{2C - k}\lambda = 0$$

or

$$(4AC - 2Ak - 2Ck + k^2)\lambda - B^2 = 0$$

or

$$[k^2 - 2(A + C)k + (4AC - B^2)]\lambda = 0 .$$

Now either λ or the coefficient must be zero. But if λ were zero, μ would also be zero, which is impossible, since (λ,μ) is a pair of direction cosines and $\lambda^2 + \mu^2 = 1$. Therefore,

$$(6) \qquad k^2 - 2(A + C)k + (4AC - B^2) = 0 .$$

Equation (6) is called the characteristic equation for the given second-degree equation and its roots are called characteristic values for the quadratic equation. We note that the sum of the roots is $2(A + C)$ while the product of the roots is $4AC - B^2$ or δ, the characteristic of the quadratic equation.

We may then solve Equation (6) for k and substitute these values in Equations (5) to determine the pairs of direction cosines (λ,μ). These pairs of values may then be substituted in (4) to obtain the equations of axes of symmetry. We note that if the characteristic is zero, Equation (6) has only one non-zero root. In Equation (5) k must be non-zero; hence, only one pair of direction cosines may be obtained, and the graph of the quadratic equation will have only one axis of symmetry. This is consistent with our previous observations that the parabola has only one axis of symmetry and that the characteristic of its equation is zero. We also note that the characteristic equation will have equal roots only if

$$(A + C)^2 = 4AC - B^2$$

or

$$A^2 + 2AC + C^2 = 4AC - B^2$$

or

$$A^2 - 2AC + C^2 = -B^2$$

or

$$(A - C)^2 = -B^2 .$$

This may only be true if B is zero and A equals C. When this is the case, you will recall that the graph of the quadratic equation is a circle. Equations (5) are satisfied by any pair of direction cosines, and there are infinitely many equations (4). This is not surprising inasmuch as every diameter of a circle determines an axis of symmetry. It is a fact that the characteristic equation of a quadratic equation always has real roots. Furthermore, if these roots determine two axes of symmetry, these axes are perpendicular. We are familiar with the fact that the intersection of two perpendicular axes of symmetry is a point of symmetry. This suggests one way to find a point of symmetry.

We may also discover points of symmetry from the definition of point of symmetry given in Section 6-8 and from the conditions on Expression (3) above:

(7) $$(2Ax' + By' + D)\lambda + (Bx' + 2Cy' + E)\mu = 0 .$$

You should recall that (x',y') is the midpoint of a chord of the graph while (λ,μ) is a pair of direction cosines in the parametric representation of the chord. When we wanted to find an axis of symmetry, λ and μ were fixed while (x',y') was variable. However, here we want to find a fixed point (x',y') which will satisfy Equation (7) for all pairs (λ,μ). This will be the case only if the coefficients of λ and μ are both zero; that is, if

(8) $$2Ax' + By' + D = 0$$

and

$$Bx' + 2Cy' + E = 0 .$$

A solution of this pair of equations will be a point of symmetry or center of the graph of the second-degree equation. The pair of equations will have a unique solution if

$$\begin{vmatrix} 2A & B \\ B & 2C \end{vmatrix} = 4AC - B^2 = \delta \neq 0 .$$

Example 1. Find the axes of symmetry and center of the graph of $8x^2 - 4xy + 5y^2 - 36x + 18y + 9 = 0$.

Solution. The characteristic equation [Equation (6)] becomes

$$k^2 - 2(8 + 5) + 4(8)(5) - (4)^2 = 0$$

or

$$k^2 - 26k + 144 = 0$$

or

$$(k - 8)(k - 18) = 0 .$$

The characteristic values are 8 and 18 . Now Equations (5) become

$$\begin{aligned} 2(8)\lambda + (-4)\mu &= 8\lambda \\ (-4)\lambda + 2(5)\mu &= 8\mu \end{aligned} \quad \text{and} \quad \begin{aligned} 2(8)\lambda + (-4)\mu &= 18\lambda \\ (-4)\lambda + 2(5)\mu &= 18\mu \end{aligned}$$

or

$$\begin{aligned} 8\lambda - 4\mu &= 0 \\ -4\lambda + 2\mu &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 2\lambda + 4\mu &= 0 \\ 4\lambda + 8\mu &= 0 . \end{aligned}$$

These pairs of equations are dependent, but since $\lambda^2 + \mu^2 = 0$, we may obtain the solutions $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

If we substitute these values in Equation (4), we obtain the equations of the axes of symmetry:

$$[2(8)\frac{1}{\sqrt{5}} + (-4)\frac{2}{\sqrt{5}}]x + [(-4)\frac{1}{\sqrt{5}} + 2(5)\frac{2}{\sqrt{5}}]y + [(-36)\frac{1}{\sqrt{5}} + 18 \cdot \frac{2}{\sqrt{5}}] = 0$$

or $8x + 16y = 0$

or $x + 2y = 0$,

and

$$[2(8)(\frac{-2}{\sqrt{5}}) + (-4)\frac{1}{\sqrt{5}}]x + [(-4)(\frac{-2}{\sqrt{5}}) + 2(5)\frac{1}{\sqrt{5}}]y + (-36)(\frac{-2}{\sqrt{5}}) + 18 \cdot \frac{1}{\sqrt{5}}] = 0$$

or $-36x + 18y + 90 = 0$

or $2x - y - 5 = 0$.

Equations (8) will enable us to find the center. The pair of equations

$$2(8)x + (-4)y + (-36) = 0$$
$$(-4)x + 2(5)y + 18 = 0$$

or

$$4x - y = 9$$
$$-4x + 10y = -18$$

has the unique solution $(2,-1)$. The point is the center or point of symmetry for the graph. We note that this point is also the intersection of the axes of symmetry.

Exercises S7-7b

Find the axes of symmetry and centers, if any, of the graphs:

1. $xy + 5x - 2y - 10 = 0$
2. $2x^2 + xy - 6y^2 + 7x - 7y + 3 = 0$

S7-8. Degenerate and Imaginary Conics and the Discriminant Δ

In our treatment of the second-degree or quadratic equation in the previous two sections, we have restricted our discussion to equations with graphs which are proper conic sections. We have made certain restrictions on the constants of the equation. In this section we shall relax these restrictions and consider the loci, if any, of the resulting equations. We shall also develop means of identifying and classifying the various possibilities. We have already encountered the *degenerate conic sections* whose graphs are single points, or pairs of lines which may be parallel, concurrent, or coincident. We have also considered equations whose loci are empty, but which are called *imaginary circles* and *imaginary ellipses* because of the form of their equations.

In Section 6-3 we have considered the problem of factoring functions. If we can factor the left member of the equation

(1) $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A , B , and C are not all zero,

into two linear factors, we would know that the graph is the union of two lines. Under what conditions is this expression factorable? You should recall that quadratic equations in a single variable often may be solved by factoring the quadratic expression into linear factors. Such an equation may always be solved by completing the square or by using the quadratic formula, which is equivalent to completing the square. In all likelihood on some occasion you have failed to detect the linear factors in the quadratic member of an equation and have resorted to the quadratic formula, only to discover that the equation really could have been solved by factoring. This suggests that the quadratic formula may be an aid in finding linear factors. In fact, the quadratic expression $ax^2 + bx + c$ may always be expressed as the product of linear factors as

$$ax^2 + bx + c = a\left(x - (\frac{-b + \sqrt{b^2 - 4ac}}{2a})\right)\left(x - (\frac{-b - \sqrt{b^2 - 4ac}}{2a})\right)$$

if we allow the use of complex numbers when necessary.

Now Equation (1) may be considered to be a quadratic equation in x if A is not zero, or in y if C is not zero. Let us assume that C is not zero and write Equation (1) as

$$Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0 , C \neq 0 . \tag{2}$$

Then

$$y = \frac{-(Bx + E) + \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2c} . \tag{3}$$

The discriminant involves terms in x^2 and x, but if it is a perfect square, we may eliminate the radical to obtain two expressions for y, say α and β, which are linear in x (i.e. α and β involve only x to the first power and various constants). Then Equation (2) and, if C is not zero, Equation (1) may be written as

$$C(y - \alpha)(y - \beta) = 0 , \tag{4}$$

where the factors of the left member are linear in x and y. The graph of Equation (4), and consequently of Equation (2), is the union of the graphs of

$$y - \alpha = 0$$
$$y - \beta = 0 ,$$

which are lines. However, the conclusion of this argument does not hold unless the discriminant of Equation (2) is a perfect square. The discriminant is

$$(Bx + E)^2 - 4C(Ax^2 + Dx + F) ,$$

as seen in Expression (3), or

(5) $$(B^2 - 4AC)x^2 + 2(BE - 2CD)x + (E^2 - 4CF) .$$

Again we make use of the quadratic formula as an aid in factoring. Expression (5) will be a perfect square if and only if the roots of the equation

(6) $$(B^2 - 4AC)x^2 + 2(BE - 2CD)x + (E^2 - 4CF) = 0$$

are equal. These roots will be equal if and only if the discriminant of Equation (6) is zero. This discriminant is

$$4(BE - 2CD)^2 - 4(B^2 - 4AC)(E^2 - 4CF) ,$$

which will be zero if and only if

$$B^2E^2 - 4BCDE + 4C^2D^2 - B^2E^2 + 4B^2CF + 4ACE^2 - 16AC^2F = 0$$

or $$-2C(2BDE - 2CD^2 - 2B^2F - 2AE^2 + 8ACF) = 0$$

or (7) $$8ACF - 2AE^2 - 2B^2F + BDE + BDE - 2CD^2 = 0$$

or $$2A(4CF - E^2) - B(2BF - DE) + D(BE - 2CD) = 0$$

or

$$2A \begin{vmatrix} 2C & E \\ E & 2F \end{vmatrix} - B \begin{vmatrix} B & D \\ E & 2F \end{vmatrix} + D \begin{vmatrix} B & D \\ 2C & E \end{vmatrix} = 0$$

or $$\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = \Delta = 0 .$$

This determinant Δ is called the discriminant of the second-degree equation. If Δ is zero, the roots of Equation (6) are equal and the Expression (5), which is the discriminant in Equation (2), is a perfect square. Thus the graph of Equation (2) is the union of two lines; if C is not zero, this set is also the graph of Equation (1).

If C is zero and A is not zero, we could go through a similar argument, treating the second-degree equation as a quadratic equation in x. Eventually we should discover that if Equation (7) holds and A is not zero, then the graph of Equation (1) is the union of two lines. But Equation (7) is equivalent to $\Delta = 0$.

If both A and C are zero, then B cannot be zero (or else the equation would no longer be of second degree), and Equation (1) reduces to

$$Bxy + Dx + Ey + F = 0 , B \neq 0 .$$

The graph will be the union of two lines if

$$Bxy + Dx + Ey + F$$

may be expressed as the product of linear factors, or as $B(x + a)(y + b)$. Now

$$Bxy + Dx + Ey + F = B(x + a)(y + b) \quad \text{for all } x \text{ and } y$$

or

$$Bxy + Dx + Ey + F = Bxy + Bbx + Bay + Bab \quad \text{for all } x \text{ and } y$$

if and only if $D = Bb$, $E = Ba$, and $F = Bab$. In this case $DE = BF$ or $BF - DE = 0$.

If A , C , and $BF - DE$ are all zero, then

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = \begin{vmatrix} 0 & B & D \\ B & 0 & E \\ D & E & 2F \end{vmatrix}$$

$$= -B \begin{vmatrix} B & D \\ E & 2F \end{vmatrix} + D \begin{vmatrix} B & D \\ 0 & E \end{vmatrix} = -B(2BF - DE) + D(BE)$$

$$= -2B^2F + BDE + BDE = -2B(BF - DE) = 0 .$$

In summary, if the graph of a second-degree equation is the union of two lines, then the discriminant is zero. The arguments which we have developed are reversible, although we have not attempted to show this here. Hence, the converse of the above is also true. If the discriminant of the general second-degree equation is zero, the left member of the equation may be expressed as the product of linear factors.

We have not considered carefully what lines, if any, these factors might represent. If Expression (5) is a perfect square, the factors are linear, but suppose that $B^2 - 4AC$, the coefficient of x^2 , is negative? We note that this is the condition when the characteristic δ is positive. In this case the coefficients in the square root are complex numbers, as are the coefficients in the linear factors. What sort of "lines" could these factors possibly represent? We shall not attempt to explore this question in detail. It is sufficient for our needs to observe that even though the coefficients are complex numbers, there still are real values which satisfy the corresponding equations. For example, the pair of equations

$$y + (2 + i)x - i = 0$$
$$y - (4 - 2i)x + 6 - 2i = 0$$

has the solution $(1,-2)$. This is always the case for the linear factors which we encounter here. The value of x which satisfies Equation (6) is real, as is the corresponding value of y . These real values are the coordinates of the point of intersection of the graphs of the corresponding linear equations. Thus, when the discriminant is zero and the characteristic is positive, the locus of a quadratic equation is a point. It is not possible that the linear factors represent dependent or inconsistent equations, for the coefficients of x and y cannot be proportional. (Why?)

If both the discriminant and the characteristic are zero, Expression (5) is a perfect square only if it reduces to $E^2 - 4CF$. (Why?) The locus of the equation will be empty, two coincident lines, or two parallel lines according as $E^2 - 4CF$ is negative, zero, or positive.

If the discriminant is zero and the characteristic is negative, we note that $E^2 - 4CF$ must be non-negative. Otherwise, Expression (5) would only be a perfect square if the coefficient of x were complex, which is impossible. The linear factors cannot represent dependent or inconsistent equations (Why?), and the locus of the second-degree equation is two intersecting lines.

Example. Find the locus of $2x^2 + xy - 6y^2 + 7x - 7y + 3 = 0$.

Solution. We determine that $\Delta = 0$, and seek to factor the left member of the equation by grouping the second-degree terms.

$$\begin{aligned} &2x^2 + xy - 6y^2 + 7x - 7y + 3 \\ &= (2x - 3y)(x + 2y) + (7x - 7y) + 3 \ . \end{aligned}$$

By inspection and trial we discover the factors

$$(2x - 3y + 1)(x + 2y + 3) \ .$$

Hence the quadratic equation may be written

$$(2x - 3y + 1)(x + 2y + 3) \ .$$

The locus of the equation is two intersecting lines. If we had not been able to find factors in this way, we could have considered the equation to be a quadratic equation in one variable, say y as above, and could have used the quadratic formula to determine the factors.

Exercises S7-8

1. Determine whether the following equations represent degenerate conic sections. If so, find the linear factors of the left member and the graph.

   (a) $6xy + 3x - 8y - 4 = 0$

   (b) $2x^2 + 8xy - x + 4y - 1 = 0$

   (c) $4x^2 - 5xy + 9y^2 - 1 = 0$

   (d) $2x^2 - xy - 6y^2 = 0$

2. If the discriminant of a second-degree equation is zero, but the characteristic is not zero, why cannot the linear factors of the left member of the equation represent dependent or inconsistent linear equations?

3. If both the discriminant and the characteristic of a quadratic equation are zero, show why Expression (5) must reduce to $E^2 - 4CF$. Why must the linear factors represent dependent or inconsistent equations?

S7-9. Invariants of the Second-Degree Equation

We have made many observations and devised several tests for the second-degree equation. We have obtained these results with the equation written in special forms. We shall show that the values of the characteristic δ and the discriminant Δ , as well as certain other algebraic expressions, are not changed by the transformations which we have used. We shall say that these values are invariant under translation and rotation of axes.

We consider a translation of axes as described in Section S7-7. If we denote the new coefficients by primes, we have

$$\begin{aligned}
A' &= A\\
B' &= B\\
C' &= C\\
D' &= 2Ah + Bk + D\\
E' &= Bh + 2Ck + E\\
F' &= Ah^2 + Bhk + Ck^2 + Dh + Ek + F \, .
\end{aligned}$$

We note that A, B, C, $A + C$, and consequently δ are invariant. To show that the discriminant is unchanged we consider

$$\Delta' = \begin{vmatrix} 2A' & B' & D' \\ B' & 2C' & E' \\ D' & E' & 2F' \end{vmatrix} = \begin{vmatrix} 2A & B & 2Ah + Bk + D \\ B & 2C & Bh + 2Ck + E \\ 2Ah + Bk + D & Bh + 2Ck + E & 2(Ah^2 + Bhk + Ck^2 + Dh + Ek + F) \end{vmatrix} .$$

We recall that adding a linear combination of several rows or columns to yet another row or column does not change the value of the determinant. We first try to make the upper right element be D. We multiply the elements of the first column by $-h$, those of the second column by $-k$, and add the sum to the third column to obtain

$$\Delta' = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ 2Ah + Bk + D & Bh + 2Ck + E & Dh + Ek + 2F \end{vmatrix} .$$

To make the lower left element be D, we multiply the elements of the first row by $-h$, those of the second row by $-k$, and add the sum to the third row. Thus

$$\Delta' = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = \Delta ,$$

and we have shown the discriminant to be invariant under translation of axes.

Now we consider a rotation of axes as described in Section S7-6. If we denote the new coefficients by primes, we have

$$\begin{aligned} A' &= A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta \\ B' &= -2A \sin \theta \cos \theta + B \cos^2 \theta - B \sin^2 \theta + 2C \sin \theta \cos \theta \\ &= B(\cos^2 \theta - \sin^2 \theta) - 2(A - C)\sin \theta \cos \theta \\ &= B \cos 2\theta - (A - C)\sin 2\theta \\ C' &= A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta \\ D' &= D \cos \theta + E \sin \theta \\ E' &= -D \sin \theta + E \cos \theta \\ F' &= F. \end{aligned}$$

In this case the coefficients in δ' and Δ' become quite complicated. We will first consider certain simpler expressions involving the coefficients. We shall then use these results to prove that δ and Δ are invariant. We note that F is invariant. $A + C$ is also invariant, for

$$\begin{aligned} A' + C' &= A(\cos^2\theta + \sin^2\theta) + C(\sin^2\theta + \cos^2\theta) \\ &= A + C \ . \end{aligned}$$

$(A - C)^2 + B^2$ is invariant, for

$$\begin{aligned} A' - C' &= (A - C)\cos^2\theta + 2B\sin\theta\cos\theta + (C - A)\sin^2\theta \\ &= (A - C)(\cos^2\theta - \sin^2\theta) + B(2\sin\theta\cos\theta) \\ &= (A - C)\cos 2\theta + B\sin 2\theta \end{aligned}$$

and

$$\begin{aligned} (A' - C')^2 + B'^2 &= (A - C)^2\cos^2 2\theta + 2B(A - C)\cos 2\theta\sin 2\theta + B^2\sin^2 2\theta \\ &\quad + B^2\cos^2 2\theta - 2B(A - C)\cos 2\theta\sin 2\theta + (A - C)^2\sin^2 2\theta \\ &= (A - C)^2(\cos^2 2\theta + \sin^2 2\theta) + B^2(\sin^2 2\theta + \cos^2 2\theta) \\ &= (A - C)^2 + B^2 \ . \end{aligned}$$

Also $D^2 + E^2$ is invariant, for

$$\begin{aligned} D'^2 + E'^2 &= D^2\cos^2\theta + 2DE\cos\theta\sin\theta + E^2\sin^2\theta \\ &\quad + D^2\sin^2\theta - 2DE\cos\theta\sin\theta + E^2\cos^2\theta \\ &= D^2(\cos^2\theta + \sin^2\theta) + E^2(\sin^2\theta + \cos^2\theta) \\ &= D^2 + E^2 \ . \end{aligned}$$

Now

$$\begin{aligned} \delta &= 4AC - B^2 \\ &= (A + C)^2 - (A - C)^2 - B^2 \\ &= (A + C)^2 - [(A - C)^2 + B^2] \ . \end{aligned}$$

Since $(A + C)^2$ and $(A - C)^2 + B^2$ are invariant, their difference, which is the characteristic, is invariant under rotation of axes.

It remains to show that the discriminant Δ is invariant under rotation. We recall from Section S7-8, Equation (7) that

$$\Delta = 8ACF - 2AE^2 - 2B^2F + 2BDE - 2CD^2 \ .$$

We rewrite this as

$$\Delta = 8ACF - 2B^2F + 2BDE - (AE^2 + AD^2 + CE^2 + CD^2) - (AE^2 - AD^2 - CE^2 + CD^2)$$

or $\Delta = 2F(4AC - B^2) + 2BDE - (A + C)(E^2 + D^2) - (A - C)(E^2 - D^2)$.

We have already noted that F , $4AC - B^2$, $A + C$, and $E^2 + D^2$ are invariant. Thus, the first and third terms are invariant. We still must show that $2BDE - (A - C)(E^2 - D^2)$ is invariant.

Now $2B'D'E' = 2[B \cos 2\theta - (A - C)\sin 2\theta](D \cos \theta + E \sin \theta)(-D \sin \theta + E \cos \theta)$

$= 2[B \cos 2\theta - (A - C)\sin 2\theta][-D^2 \sin \theta \cos \theta + E^2 \sin \theta \cos \theta + DE(\cos^2 \theta - \sin^2 \theta)]$

$= [B \cos 2\theta - (A - C)\sin 2\theta][(E^2 - D^2)(2 \sin \theta \cos \theta) + 2DE(\cos^2 \theta - \sin^2 \theta)]$

$= [B \cos 2\theta - (A - C)\sin 2\theta][(E^2 - D^2)\sin 2\theta + 2DE \cos 2\theta]$,

$E'^2 - D'^2 = (-D \sin \theta + E \cos \theta)^2 - (D \cos \theta + E \sin \theta)^2$

$= D^2 \sin^2 \theta - 2DE \sin \theta \cos \theta + E^2 \cos^2\theta - D^2 \cos^2 \theta - 2DE \sin \theta \cos \theta - E^2 \sin^2\theta$

$= (E^2 - D^2)(\cos^2 \theta - \sin^2 \theta) - 2DE(2 \sin \theta \cos \theta)$

$= (E^2 - D^2)\cos 2\theta - 2DE \sin 2\theta$,

and

$$A' - C' = (A - C)\cos 2\theta + B \sin 2\theta \ .$$

Thus,

$2B'D'E' - (A' - C')(E'^2 - D'^2) = [B \cos 2\theta - (A - C)\sin 2\theta][(E^2 - D^2)\sin 2\theta + 2DE \cos 2\theta]$

$\qquad - [B \sin 2\theta + (A - C)\cos 2\theta][(E^2 - D^2)\cos 2\theta - 2DE \sin 2\theta]$

$= \cos^2 2\theta[2BDE - (A - C)(E^2 - D^2)]$

$\qquad + \sin^2 2 [-(A - C)(E^2 - D^2) + 2BDE]$

$\qquad + \sin 2\theta \cdot \cos 2\theta[B(E^2 - D^2) - (A - C)(2DE) - B(E^2 - D^2) + (A - C)(2DE)]$

$= (\sin^2 2\theta + \cos^2 2\theta)[2BDE - (A - C)(E^2 - D^2)]$

$= 2BDE - (A - C)(E^2 - D^2)$.

Thus the discriminant of the second-degree equation is also invariant under rotation.

We note that if the graph of the second-degree equation has a point of symmetry, or represents a central conic, then after a translation of the axes which makes the new origin the point of symmetry, the new equation is

$$A'x^2 + B'xy + C' + F' = 0 \ ,$$

for which

$$\Delta' = \begin{vmatrix} 2A' & B & 0 \\ B & 2C' & 0 \\ 0 & 0 & 2F' \end{vmatrix}$$

$$= 2F' \begin{vmatrix} 2A' & B \\ B & 2C' \end{vmatrix}$$

$$= 2F'\delta' .$$

$$F' = \frac{\Delta'}{2\delta'} ,$$

but since Δ and δ are invariant under translation,

$$F' = \frac{\Delta}{2\delta} ,$$

and the transformed equation is

$$Ax^2 + Bxy + Cy^2 + \frac{\Delta}{2\delta} = 0 .$$

S7-10. Summary

We have shown that if the locus of a second-degree equation is not empty, then the graph is either a proper conic section or a degenerate conic section. We have developed many methods and criteria for analyzing such equations and have found certain invariants called the characteristic and discriminant particularly important. We summarize some of these results in the form of a table.

	$\delta < 0$	$\delta = 0$	$\delta > 0$
$\Delta = 0$	intersecting lines	empty, or parallel or coincident lines	point-ellipse or point-circle.
$\Delta \neq 0$	hyperbola	parabola	circle, ellipse, or empty

Example. Discuss the locus of

$$8x^2 - 4xy + 5y^2 - 36x + 18y + 9 = 0 .$$

Solution. Here $\Delta = -10{,}368$ and $\delta = 144$.

Since $B \neq 0$, the locus may not be a circle, but may be an ellipse.

$$F' = \frac{\Delta}{2\delta} = -36 ,$$

so the locus is a real ellipse.

If we substitute coefficients in the equations

$$2Ah + Bk + D = 0$$
$$Bh + 2Ck + E = 0 \ ,$$

we obtain

$$16h - 4k - 36 = 0$$
$$-4h + 10k + 18 = 0 \ ,$$

which give $(2,-1)$ as the center of the ellipse.
The characteristic equation

$$k^2 - 2(A + C)k + (4AC - B^2) = 0$$

is

$$k^2 - 26k + 144 = 0 \ ,$$

which gives 8 and 18 as the characteristic values.
These are substituted in the equations

$$2A\lambda + B\mu = k\lambda$$
$$B\lambda + 2C\mu = k\mu$$

to obtain

$$8\lambda - 4\mu = 0 \qquad \text{and} \qquad 2\lambda + 4\mu = 0$$
$$[illegible]\lambda + 2\mu = 0 \qquad \qquad 4\lambda + 8\mu = 0 \ ,$$

which give $\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)$ as pairs of direction cosines for the axes of symmetry

$$(2A\lambda + B\mu)x + (B\lambda + 2C\mu)y + (D\lambda + E\mu) = 0$$

or

$$x + 2y = 0$$
$$2x - y - 5 = 0 \ .$$

The translation of axes gives the equation

$$8x^2 - 4xy + 5y^2 - 36 = 0 \ ,$$

while the rotation of axes through an angle θ such that $\tan 2\theta = \frac{B}{A - C}$, gives the transformed equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \ .$$

Primes have been omitted consistently in the interest of simplicity.

Exercises 8-10

Identify the graphs of the following equations. Obtain the transformed equation reduced to standard form. Sketch the graph, locating the center (if any) and indicate axes of symmetry.

1. $8x^2 - 12xy + 17y^2 - 20 = 0$
2. $3x^2 + 12xy - 13y^2 - 135 = 0$
3. $5x^2 - 6xy + 5y^2 - 16x + 16y + 8 = 0$
4. $9x^2 - 24xy + 16y^2 - 20x - 15y = 0$
5. $9x^2 - 24xy + 16y^2 + 60x - 80y + 100 = 0$
6. $3x^2 + 10xy + 3y^2 + 16x + 16y + 24 = 0$
7. $5x^2 + 6xy + 5y^2 - 16x - 16y + 8 = 0$
8. $27x^2 - 48xy + 13y^2 - 12x + 44y - 77 = 0$
9. $12x^2 - 7xy - 12y^2 - 41x + 38y + 22 = 0$
10. $13x^2 + 48xy + 27y^2 + 44x + 12y - 77 = 0$
11. $9x^2 - 24xy + 16y^2 + 90x - 120y + 200 = 0$
12. $10xy + 4x - 15y - 6 = 0$

Supplement to Chapter 10

GEOMETRIC TRANSFORMATIONS

S10-1. Isometries of the Line

In previous chapters we have seen examples of mappings of a line onto a line and of a plane onto a plane. Some of these had the property of preserving the distance between any two points and are therefore called "isometries," (from Greek, ισος meaning same and μετρειν meaning to measure). Therefore, an isometry, having this property, will map any configuration onto a congruent configuration. In fact this amounts to a definition of congruence. In this chapter we want to investigate the isometries of the line and of the plane and consider other types of mappings or transformations.

Let us consider in more generality the isometric transformations of a line. Each point P with coordinate x will be mapped onto its image point P' with coordinate $x' = f(x)$. Furthermore, for any two points with coordinates x_1 and x_2, we have

$$(1) \qquad |x_1 - x_2| = |f(x_1) - f(x_2)| .$$

We distinguish two cases according as the origin is a fixed point or is not a fixed point.

If zero is a fixed point, we have $f(0) = 0$, so that with $x_2 = 0$, (1) becomes

$$|x_1 - 0| = |f(x_1) - f(0)|$$

or

$$|x| = |f(x)| .$$

This implies that either $f(x) = x$ or $f(x) = -x$. In the former, each point is mapped onto itself and this is called the identity transformation I. In the latter we have a transformation which can be described as a reflection in the point 0, because each point is mapped onto its mirror-like image with respect to 0.

If zero is not a fixed point, it is mapped onto some point with a nonzero coordinate and we can write $f(0) = a \neq 0$. Thus with $x_2 = 0$, (1) becomes

$$|x_1 - 0| = |f(x_1) - f(0)|$$

or

$$|x| = |f(x) - a| .$$

This implies that either $f(x) - a = x$ or $f(x) - a = -x$. The former is $f(x) = x + a$ which is a translation and the latter is $f(x) = -x + a$. The transformation represented by $f(x) = -x + a$ can be described by saying that the image of any point is obtained by a reflection in the origin followed by a translation of a. We now have

THEOREM S10-1. An isometry of the line is either

(1) the identity transformation
(2) a translation
(3) a reflection in the origin
or (4) a reflection in the origin followed by a translation;
and conversely.

The fourth possibility in Theorem S10-1 raises the general question of one transformation followed by another. If the first transformation is f and the second is g, we define the product or composite transformation to be the transformation

$$gf : x \rightarrow x' = g[f(x)] ,$$

where $x \rightarrow x'$ means that the image of x under the mapping gf is x'. As we have seen, the transformation $x \rightarrow -x + a$ is a composite of $f(x) = -x$ followed by $g(x) = x + a$ since $g[f(x)] = -x + a$. From the definition of an isometry, it seems reasonable to expect that the product of two isometries should be an isometry. We show this to be true in the following case.

Example. Show that the translation $f(x) = x + a$ followed by the translation $g(x) = x + b$ is an isometry.

Solution. We have

$$g[f(x)] = (x + a) + b = x + (a + b)$$

which represents a translation. Thus the composite transformation is an isometry.

Exercises S10-1

1. By considering the remaining possibilities in similar fashion, show that the composite of any two isometries of the line is again an isometry.
2. Prove the converse of Theorem S10-1.

In the first exercise above, it was necessary to consider a translation followed by a reflection. If $g(x) = x + a$ is followed by $f(x) = -x$, the composite transformation is

$$fg : f[g(x)] = -(x + a) = -x - a .$$

This is certainly an isometry since it is a reflection followed by a translation $-a$. We see that composition of transformations is not necessarily commutative since in this case $fg \neq gf$. However we can generate any isometry by an appropriate sequence of compositions using only translations and reflections. It is not difficult to show that the isometries of a line form a group since the operation of composition is associative and to each isometry f, there exists an inverse isometry f^{-1} such that $f^{-1}f = I$. As we have observed, this group is non-commutative.

S10-2. Isometries of the Plane

In previous chapters we considered two changes of coordinate systems in the plane called translation and rotation. The same effect can be produced by mappings of the plane onto itself, which leave the coordinate axes unchanged. The contrast to this is the previous approach in which the plane remained fixed and the coordinate axes were changed.

In this context, a translation is a mapping of the form

$$(x,y) \longrightarrow (x',y') = (x+h\ ,\ y+k)\ .$$

A rotation is a mapping in which each point is mapped onto a point the same distance from the origin. These points determine rays from the origin which form an angle in standard position whose measure is increased by θ .

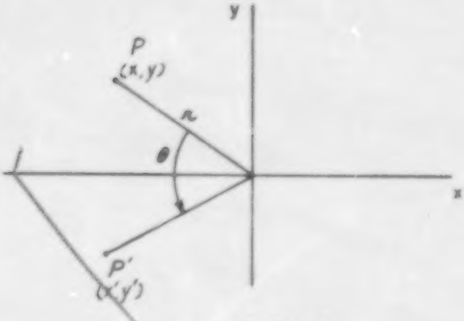


Figure S10-1

Let (r,ϕ) be a point P described in polar coordinates where the polar axis is the positive side of the x-axis. The rotation mapping can now be written as

$$(r,\phi) \longrightarrow (r,\phi+\theta)\ .$$

In terms of rectangular coordinates, we have

$$\begin{aligned} x' = r\cos(\phi+\theta) &= r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x\cos\theta - y\sin\theta\ . \\ y' = r\sin(\phi+\theta) &= r\sin\phi\cos\theta + r\cos\phi\sin\theta \\ &= x\sin\theta + y\cos\theta\ . \end{aligned}$$

The proofs that these mappings are isometries are left as exercises.

The previous discussion of reflection with respect to a point can be extended to the plane. A reflection in the origin can be defined by the transformation

$$(x,y) \rightarrow (x',y') = (-x,-y) \ .$$

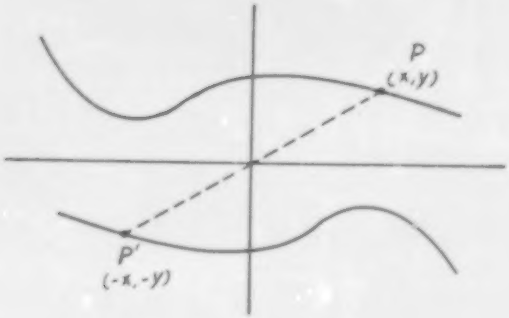


Figure S10-2

The description of this transformation is particularly simple in terms of polar coordinates since $P(r,\theta) \rightarrow P'(-r,\theta)$. By using the distance formula for the appropriate coordinate system, it is easy to verify that this transformation is an isometry of the plane. However a rotation of π radians is the same transformation. This can be seen by letting $\theta = \pi$ in the rectangular description of a rotation to obtain

$$x' = x \cos \pi - y \sin \pi = -x$$
$$y' = x \sin \pi + y \cos \pi = -y \ ,$$

or by letting $\theta = \pi$ in the polar description to obtain

$$(r,\theta) \rightarrow (r \ , \theta + \pi) \ .$$

The last ordered pair represents a point in polar coordinates which can also be represented as $(-r,\theta)$.

We now introduce another transformation which can be described as a reflection in a line. The image of a point is found by constructing a perpendicular to the line and extending it on the other side a distance equal to the distance of the point from the line.

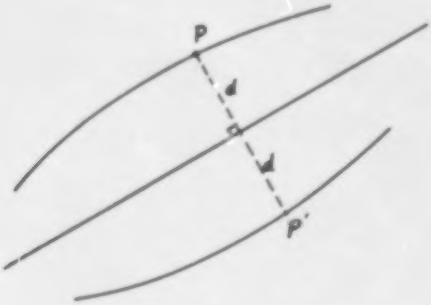


Figure S10-3

The transformation equations for reflections in certain lines can be written down immediately. For instance, for reflection in the x-axis, we have $(x,y) \longrightarrow (x',y') = (x,-y)$

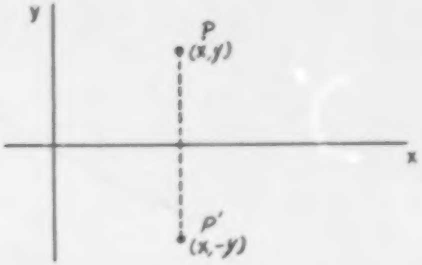


Figure S10-4

For reflection in the y-axis, we have

$$(x,y) \rightarrow (x',y') = (-x,y) \ .$$

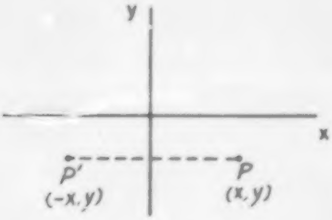


Figure S10-5

We can similarly define the product of two transformations of the plane onto itself, and again would expect the product of two isometries to be an isometry. In fact we will show that any isometry of the plane can be described solely in terms of reflections. Thus the group of isometries of the plane with composition can be generated from the set of reflections alone.

Example. Find the isometry composed of reflection in the line $x = 1$ followed by reflection in the line $x = 4$.

Solution.

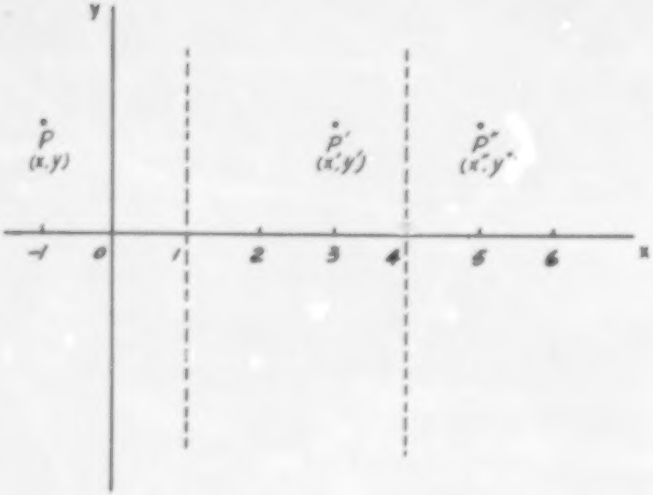


Figure S10-6

The first reflection maps $P(x,y)$ onto $P'(x',y') = (-x + 2\ ,\ y)$ and the second maps (x',y') onto $(x'',y'') = (-x' + 8\ ,\ y')$. By composition of the mappings, we immediately have

$$x'' = -x' + 8 = -(-x + 2) + 8 = x + 6$$
$$y'' = y' = y$$

and we recognize these as the equations of a translation which maps each point onto the point six units to the right.

Exercises S10-2

1. Do the two mappings in the example commute under composition?
2. Find the equations to describe the mapping of reflection in an arbitrary vertical line $x = h$ and in an arbitrary horizontal line $y = k$.
3. Using Exercise 2 find the composite mapping given by successive reflection in either 2 horizontal or 2 vertical lines.
4. What is the composite mapping given by reflection in the line $x = h$ followed by reflection in the line $y = k$?
5. Do the mappings in Exercises 3 and 4 commute under composition?

S10-3. Reflections and Isometries

The above exercises illustrate the proposition that any translation or any reflection in a point can be obtained by a succession of reflections in appropriate lines. We observed previously that a reflection in O is equivalent to a rotation of π radians, so that a rotation of π radians can be obtained by a succession of reflections. Let us try to establish further connections between reflections and rotations by describing a reflection in a line L in terms of polar coordinates. Choose the pole of the coordinate system on the line in which the reflection is to be made and let the equation of the line L be $\theta = k$, a constant.

From Figure S10-7 it can be seen that $r' = r$ and that a measure of ϕ' is $\theta + (\theta - \phi) = 2\theta - \phi$ for this particular diagram. We can show this in general if we start with the angle 2θ and subtract the angle ϕ to arrive at the terminal side of the angle ϕ'. Thus the reflection in the line L is the mapping

(1) $$R_L : (r,\phi) \to (r',\phi') = (r\ ,\ 2\theta - \phi)$$

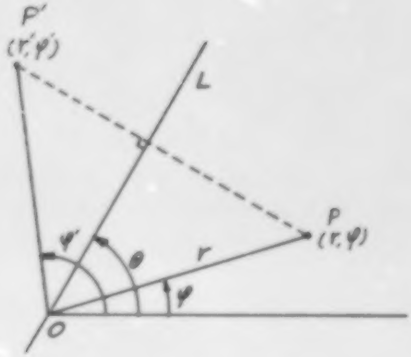


Figure S10-7

Suppose we now carry out successive reflections in two lines L and M through O with equations $\phi = \theta_1$. and $\phi = \theta_2$. By (1) we can denote the reflections by

$$R_L : (r,\phi) \rightarrow (r',\phi') = (r\ ,\ 2\theta_1 - \phi)$$

$$R_M : (r',\phi') \rightarrow (r'',\phi'') = (r'\ ,\ 2\theta_2 - \phi')\ .$$

The composite transformation R_L followed by R_M can be described as

$$R_M R_L : (r,\phi) \rightarrow (r'',\phi'')$$

where $$r'' = r' = r$$

and $$\phi'' = 2\theta_2 - \phi' = 2\theta_2 - (2\theta_1 - \phi) = \phi + 2(\theta_2 - \theta_1)\ .$$

We recognize this as the description of a rotation of $2(\theta_2 - \theta_1)$; thus, the composite mapping of two reflections in intersecting lines is a rotation.

Exercises S10-3

1. By reversing the above argument, prove that any rotation is the product of line reflections.
2. Using the notation of the preceding discussion, determine $R_L R_M$.

We are now in a position to prove

THEOREM S10-2. Any isometry of the plane is composed of at most three line reflections.

Proof. Assume we have some distance-preserving transformation which will therefore map an arbitrary triangle ABC onto a congruent triangle A'B'C'. The line through the points A and B may or may not intersect the line through the points A' and B'. Hence we consider two cases.

Case I. The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect.

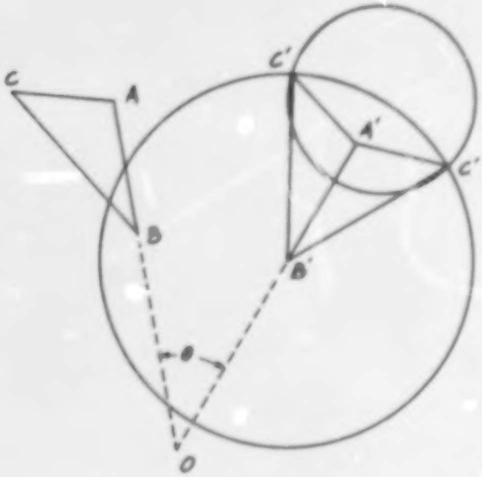


Figure S10-8

From Figure S10-8 we see there are two possible positions for the point C' at points of intersection of the circles given by the conditions d(A',C') = d(A,C) and d(B',C') = d(B,C). For one position of C', the transformation is a rotation θ about O, which can be represented as the product of two line reflections. For the other position of C', the transformation is the same rotation followed by a reflection in the line through A' and B', and therefore is the product of three line reflections.

Case II. The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel.

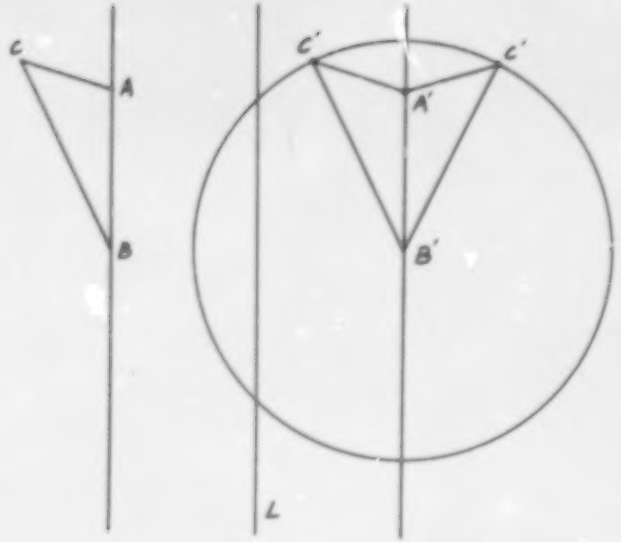


Figure S10-9

Once again there are two possible positions for the point C'. Consider the line L midway between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. Then for one position of C', the transformation is a reflection in L. For the other position of C', the transformation is a reflection in L followed by a reflection in the line $\overleftrightarrow{A'B'}$, which completes the proof of the theorem.

S10-4. Non-isometric Transformations

In Section S2-2, in addition to the transformations of a line onto itself called translation and reflection, the transformation's expansions and contractions were defined. An expansion is a mapping $x \to x' = ax$ where $a > 1$ and a contraction is a mapping $x \to x' = ax$ where $0 < a < 1$. It is apparent that neither of these is an isometry since the origin is mapped onto itself and the point whose coordinate is 1 is mapped onto the point whose coordinate is a , but $|1 - 0| \neq |a - 0|$. We may consider the compositions of these transformations with themselves and with isometries to obtain a general class of transformations of the form

$$x \rightarrow x' = ax + b \quad a \neq 0 .$$

known as the class of linear transformations. As we noted in Section S2-2, this set of linear transformations with the operation of composition forms a group.

The idea of a linear transformation extends naturally to the plane by considering the mapping $(x,y) \rightarrow (x',y')$ where

$$x' = ax + by + h \ , \ |a| + |b| \neq 0 \ ,$$

$$y' = cx + dy + k \ , \ |c| + |d| \neq 0 \ ,$$

We see immediately that this mapping is the composition of the mapping

$$(x,y) \rightarrow (x',y') = (ax + by \ , \ cx + dy)$$

followed by the translation

$$(x',y') \rightarrow (x'',y'') = (x' + h \ , \ y' + k) \ .$$

Therefore we consider a subset of the set of linear transformations of the plane, namely those transformations of the form $(x,y) \rightarrow (x',y') = (ax + by \ , \ cx + dy)$ which leave the origin fixed. This subset includes the rotations and reflections in the plane previously discussed in this chapter. One of the things that can be done in general with this subset is to investigate whether it forms a group under composition. The identity mapping is an identity element for the operation of composition. Hence a given mapping will have an inverse if it can be followed by a mapping which will map (x',y') back onto (x,y) . To find whether such a mapping exists, we consider the composite mapping $(x,y) \rightarrow (x',y') = (ax + by \ , cx + dy)$ followed by $(x',y') \rightarrow (x'',y'') = (px' + qy' \ , \ rx' + sy)$. We obtain the mapping $(x,y) \rightarrow (x'',y'')$ where

$$\begin{aligned} x'' &= p(ax + by) + q(cx + dy) = (ap + cq)x + (bp + dq)y \\ y'' &= r(ax + by) + s(cx + dy) = (ar + cs)x + (br + ds)y \ , \end{aligned}$$

which is a mapping of the same form. Thus, given a , b , c , d , we want to determine p , q , r , s so that the composite mapping is the identity mapping; that is, so that

$$\begin{aligned} ap + cq &= 1 \\ bp + dq &= 0 \\ ar + cs &= 0 \\ br + ds &= 1 \ . \end{aligned}$$

This is actually two linear systems, each consisting of two equations in two unknowns, which can be solved to obtain

$$p = \frac{d}{ac - bc}, \quad q = \frac{-b}{ad - bc}, \quad r = -\frac{c}{ad - bc}, \quad s = \frac{a}{ac - bc},$$

if $ad - bc \neq 0$. Thus, a mapping will have an inverse if and only if $ad - bc \neq 0$. It is left as an exercise to prove that this set of transformations is associative. We combine these results in

THEOREM S10-3. The set of linear transformations of the form

$$(x,y) \rightarrow (x',y') = (ax + by\ ,\ cx + dy)$$

where $ad - bc \neq 0$, forms a group under the operation of composition.

We now consider examples of linear transformations which are not isometries.

Example 1. Discuss the linear transformation

$$(x,y) \rightarrow (x',y') = (2x + 3y\ ,\ x - y)\ .$$

Discussion. We start by examining what happens to points on certain lines under this transformation. For instance, a point on the x-axis, $(a,0)$, is mapped onto the point $(2a,a)$, which lies on the line $y = \frac{1}{2}x$. A point on the y-axis, $(0,a)$, is mapped onto the point $(3a,-a)$, which lies on the line $y = -\frac{1}{3}x$. If a point lies on a line whose equation is $ax + by + c = 0$, we can find a condition on the coordinates of its image by expressing x and y in terms of x' and y' and substituting in the equation. From the equations of the transformation we get

$$x = \frac{1}{5}(x' + 3y')$$

$$y = \frac{1}{5}(x' - 2y).$$

(This also shows that any point (x',y') is the image of some (x,y). Thus a point on the line is mapped onto a point (x',y') such that

$$a(x' + 3y') + b(x' - 2y') + 5c = 0$$

or

$$(a + b)x' + (3a - 2b)y' + 5c = 0\ .$$

which is an equation of a line. Hence a line is mapped onto a line, and if the line contains the origin (i.e., $c = 0$), so does its image. The images of other loci can be similarly determined.

Example 2. Discuss the linear transformation

$$(x,y) \longrightarrow (x',y') = (x + y , 2x + 2y).$$

Discussion. We first observe that this transformation does not belong to the group described in Theorem S10-3 since $1 \cdot 2 - 1 \cdot 2 = 0$. Hence it does not possess an inverse mapping under composition. We investigate this transformation geometrically. A point (a,b) is mapped onto the point $(a + b , 2a + 2b)$. This image lies on the line $y = 2x$, so that the plane is mapped onto a single line in the plane. Furthermore, infinitely many points in the plane are mapped onto each point on the line $y = 2x$. Thus the mapping does not have an inverse mapping in the sense of assigning a unique pre-image to each image point.

Since there is a one-to-one correspondence between points in the plane and complex numbers, it is not surprising that mappings of the plane can be related to complex numbers. Recall that if we have a rectangular coordinate system, this correspondence is established by associating the point (a,b) and the complex number $a + bi$. Thus any of the mappings we have discussed so far can be considered as mappings of the set of complex numbers into itself. That is, if (x,y) is mapped onto (x',y') , we consider the complex number $x + yi$ mapped onto the complex number $x' + y'i$. Since functions are mappings, functions whose domain and range are the set of complex numbers give a mapping of the set of complex numbers into itself. For example consider the function defined by $f(z) = 2z$, or the mapping $z \rightarrow z' = 2z$, where $z = x + yi$ and $z' = x' + y'i$. This function maps $x + yi$ onto $2x + 2yi$, which corresponds to mapping the point (x,y) onto the point $(x',y') = (2x,2y)$. An investigation of this mapping is left for an exercise.

We give another example of this relationship.

Example 3. Discuss the mapping defined by the equation $f(z) = z^2$.

Discussion. From the equation we have

$$z' = x' + y'i = z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi .$$

Hence, in terms of coordinates the mapping is the non-linear transformation

$$x' = x^2 - y^2$$
$$y' = 2xy .$$

We see from these equations that the hyperbola $x^2 - y^2 = k$ is mapped onto the line $x' = k$ and the hyperbola $2xy = k$ is mapped onto the line $y' = k$, (It is convenient to think of the functions as a mapping of the z-plane, with x and y coordinates, into the z'-plane, with x' and y' coordinates.) We also have

$$x'^2 + y'^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = (x^2 + y^2)^2$$

so that the circle $x^2 + y^2 = r^2$ in the z-plane is mapped onto the circle $x'^2 + y'^2 = r^4$ in the z'-plane. We see that in trying to develop a geometric description of a mapping, it is sometimes more fruitful to discuss the images of certain loci rather than the images of individual points. This mapping is an example of an important class of functions of z known as conformal mappings which have the property of preserving the angle of intersection of two curves. This property is of fundamental importance in the general theory of functions of a complex variable z. In particular, polynomials in z and their quotients will provide conformal mappings.

Sometimes information about a mapping can be obtained by using the polar representation of a complex number. Thus, if θ is the angle in standard position which contains (x,y) on its terminal side, we can write

$$z = x + yi = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2}$. De Moivre's Theorem gives us

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta) .$$

Thus, in the mapping $z \rightarrow z' = z^2$, the point (r,θ) is mapped onto the point $(r^2,2\theta)$, which gives a general geometric description of the mapping.

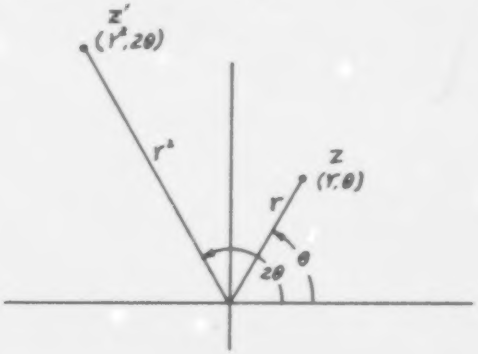


Figure S10-10

Exercises S10-4

1. Show that any transformation belonging to the group in Theorem S10-3 will map a line onto a line.

2. Discuss the transformations $(x,y) \to (2x,2y)$, $(x,y) \longrightarrow (\frac{1}{2}x, \frac{1}{2}y)$, and $(x,y) \to (2x,3y)$ by finding the image of $x^2 + y^2 = 1$.

3. In Example 2, find those points which are mapped onto the same point on $y = 2x$.

4. Show that the angle between two lines through the origin is preserved under the mapping $z \to z' = kz$.

5. Discuss the mapping $z \to z' = \frac{1}{z}$.

6. Find various equations to represent the mapping called "inversion in a circle," in which a point at distance d from the origin is mapped onto the point at distance $\frac{1}{d}$ from the origin lying on the same ray from the origin. The origin is mapped onto itself.

7. Prove that the set of linear transformations $(x,y) \longrightarrow (x',y') = (ax + by, cx + dy)$ is associative.

S10-5. Matrix Representation of Transformations

In the previous section we saw that the product of two linear transformations is again a linear transformation. It is convenient to introduce a notation to represent a linear transformation

$$x' = ax + by$$
$$y' = cx + dy .$$

Since the coefficients of x and y determine the mapping, we represent the mapping by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a matrix in general is simply a rectangular array of numbers arranged in rows (horizontally) and columns (vertically). The composite mapping fg is the mapping g followed by the mapping f . Thus, as we saw in the previous section, the mapping whose matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

followed by the mapping whose matrix is

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

is the mapping whose matrix is

$$\begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}.$$

Hence, it is natural to define a binary operation on these matrices as follows.

DEFINITION. (*Matrix multiplication*)

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}.$$

Observe that each entry in the product matrix is the inner product of a row in the left factor by a column in the right factor. Because of this, matrix multiplication can be described as "row into column" multiplication.

Example 1. Find the matrix which represents a mapping described by rotation θ.

Solution. The equations of this mapping using rectangular coordinates are

$$x' = x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta .$$

The corresponding matrix is

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} .$$

Example 2. Using matrices find the mapping composed of a reflection in the x-axis followed by a reflection in the y-axis.

Solution. As we have seen, the equations for a reflection R_x in the x-axis are

$$x' = x = 1 \cdot x + 0 \cdot y$$
$$y' = -y = 0 \cdot x + (-1)y$$

so that the corresponding matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The equations for a reflection R_y in the y-axis are

$$x' = -x = (-1)x + 0 \cdot y$$
$$y' = y = 0 \cdot x + 1 \cdot y$$

with matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix for the composite mapping $R_y R_x$ is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which corresponds to the mapping $(x,y) \to (-x,-y)$. This, as we have seen, is a reflection in O or a rotation of π radians.

Exercises S10-5a

(Use Matrices)

1. Using the notation of the example above, find $R_x R_y$.

2. Find the matrix for

   (a) reflection in the line $y = x$.

   (b) reflection in the line $y = -x$.

3. Find the matrix for, and interpret geometrically, a reflection in the line $y = x$ followed by a rotation of $\frac{\pi}{2}$ radians.

4. Describe the mapping which results from a rotation θ_1 followed by a rotation θ_2.

5. Show that matrix multiplication is associative but not commutative.

*6. Show that the matrix for a reflection in a line through O with inclination θ is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

(Hint: While this can be done directly in rectangular coordinates using trigonometry, it is also interesting to solve the problem using polar coordinates.) Verify that this matrix includes the previously discussed cases $\theta = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, and $\frac{3\pi}{4}$ radians.

7. Find the matrix for a reflection in a line through 0 with inclination θ_1 followed by a reflection in a line through 0 with inclination θ_2. Show that the answer agrees with previous results.

We have a one-to-one correspondence between two-by-two matrices (2 rows and 2 columns) and linear transformations of the plane which leave the origin unchanged. We also see, by the definition of matrix multiplication, that the product of two matrices corresponds to the mapping composed of the mappings corresponding to the matrices. Thus, the two systems are isomorphic in the sense that any operations on mappings can also be interpreted in terms of operations on the corresponding matrices. Hence Theorem S10-3 has an analogue for matrices as follows.

THEOREM S10-4. The set of matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $ad - bc \neq 0$, forms a group under the operation of matrix multiplication.

The number $ad - bc$ is called the determinant of the matrix and the matrix is called non-singular if $ad - bc \neq 0$. Thus the set in the theorem is the set of non-singular two-by-two matrices.

Since we found the inverse of a mapping in the proof of Theorem S10-3, we may write the inverse of a non-singular matrix under matrix multiplication as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$$

We now consider the matrices of isometries of the plane which leave the origin fixed. By Theorem S10-2, any such matrix is the product of at most three matrices each of which represents a reflection. By Problem 6 in Exercises S10-5a, the matrix of a reflection in a line through the origin can be written as

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

for some α. By Problem 7 in the same set of exercises, the matrix for the product of two line reflections, which is a rotation, can be written as

$$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

for some β. Since matrix multiplication is associative, the matrix for the product of three line reflections can be written as a reflection matrix times a rotation matrix; that is, as

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

for appropriate α and β. This product is

$$\begin{pmatrix} \cos\alpha\ \cos\beta + \sin\alpha\ \sin\beta & \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta \\ \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta & -\cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha - \beta) & \sin(\alpha - \beta) \\ \sin(\alpha - \beta) & -\cos(\alpha - \beta) \end{pmatrix}.$$

Thus we have the following theorem.

THEOREM S10-5. Any isometry of the plane with O as a fixed point can be represented by one of the matrices

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ or } \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha, & \cos\alpha \end{pmatrix}$$

for suitable α.

Corollary S10-5-1. The determinant of a matrix which represents an isometry of the plane with O as a fixed point is either 1 or -1.

Let s be the set of matrices which can be written in either of the forms

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \text{ or } \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

for some α. We define two matrices to be equal if and only if they are identical, that is, if and only if their corresponding entries are equal. Thus the same matrix may arise from different values of α, but we consider the matrices themselves and not the values of α . As we have seen, each such matrix represents an isometry (either a line reflection or a rotation), and by Theorem S10-5, any isometry with O a fixed point can be represented by such a matrix.

The set s forms a group, under the operation of matrix multiplication, which is a subgroup of the group described in Theorem S10-4.

Exercises S10-5b

1. Prove that the set s , just described, is a group.
2. Show that the determinant of the product of two square matrices of order 2^2 equals the product of their determinants.
3. Show that there exist matrices with determinant 1 or -1 which do not represent isometries.
4. Prove, using the distance formula, that an isometry with O as a fixed point has a matrix whose determinant is 1 or -1 .
5. Any matrix in the set s , in addition to having determinant ± 1 , has the property that the sum of the squares of the elements in any row or in any column is 1 . Prove that if a matrix has determinant ± 1 and has the sum of the squares of elements in each column (or in each row) equal to 1 , then it is a member of s .

S10-6. Symmetry

The symmetries of a geometric figure can be interpreted very nicely in terms of mappings. If a figure is mapped into itself by a particular isometry, then it has the particular kind of symmetry described by the isometry. Thus a figure may have symmetry with respect to a point if it is mapped into itself upon reflection in that point; it may have symmetry with respect to a line if it is mapped into itself upon reflection in that line. The algebraic tests for symmetry arise from the equations of the various mappings.

As you have seen, it is possible to simplify the equations of various loci by using appropriate transformations. In particular it is possible to eliminate by translation the terms involving x and y in an equation for an ellipse or a hyperbola. Then a suitable rotation will eliminate the xy term. Geometrically, what this last step involves is the determination of a suitable rotation so that the x and y axes become axes of symmetry of the figure.

We now want to solve this problem by means of the algebra of matrices. We assume that a suitable translation has been made so that the hyperbola or ellipse has its center at the origin of a rectangular coordinate system. Hence its equation is

$$f(x,y) = Ax^2 + Bxy + Cy^2 = D \ . \tag{1}$$

We want to determine a rotation so that the points which satisfy $f(x,y) = D$ will be mapped onto points which satisfy some equation not having an xy term. Since the constant term is unaffected by a rotation, we consider only the quadratic portion $f(x,y)$ of (1). If we extend our notion of matrix multiplication slightly, we can get a matrix representation of $f(x,y)$. We introduce matrices with one row and two columns or two rows and one column and define a product of one of these times a two-by-two matrix.

DEFINITION.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$(x \quad y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}$$

$$(p \quad q)\binom{r}{s} = (pr + qs)$$

Notice that the one-by-two (or two-by-one) matrices must occur in the proper position but that the multiplication is still row into column multiplication. We now associate with $f(x,y)$ the matrix

$$\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}$$

and verify without difficulty that

$$f(x,y) = (x \quad y)\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

We can similarly express a rotation θ as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} . \tag{3}$$

By Theorem S10-4 this rotation matrix has an inverse. The Equation (3) is a statement of equality of matrices and hence each member can be multiplied on the left by the inverse matrix to obtain equal matrices.

$$(4) \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(This assumes associativity of a matrix product involving non-square matrices and the proof of this is left as an exercise.)

From the definition it is not hard to see that if

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

then

$$\begin{pmatrix} ax + by & cx + dy \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Thus from (4) we have

$$\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x' & y' \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Substituting (4) and (5) into (2) we see that the rotation will transform $f(x,y)$ into

$$g(x',y') = (x'y')\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= (x'y')\begin{pmatrix} A\cos\theta - \frac{B}{2}\sin\theta & \frac{B}{2}\cos\theta - C\sin\theta \\ A\sin\theta + \frac{B}{2}\cos\theta & \frac{B}{2}\sin\theta + C\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= (x'y')\begin{pmatrix} A' & \frac{B'}{2} \\ \frac{B'}{2} & C' \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = A'x'^2 + B'x'y' + C'y'^2.$$

We now want to determine θ so that $g(x',y')$ remains unchanged when x' is replaced by $-x'$ and also when y' is replaced by $-y'$. This will occur if $g(x',y')$ does not have an $x'y'$ term.

The coefficient of $x'y'$ in $g(x',y')$ is

$$B' = 2\{(A - C)\sin\theta\cos\theta - \frac{B}{2}(\sin^2\theta - \cos^2\theta)\}$$
$$= (A - C)\sin 2\theta + B\cos 2\theta.$$

Thus

$$B' = 0 \text{ if}$$

$$\theta = \frac{\pi}{4} \text{ radians, when } A = C$$

$$\theta = \frac{1}{2} \text{ arc tan } \frac{B}{C - A}, \text{ when } A \neq C .$$

In Chapter S7 the latter angle θ was $\frac{1}{2}$ arc tan $\frac{B}{(A - C)}$, since there, the axes were rotated, whereas in this treatment, the axes remain fixed and the plane is mapped onto itself. The calculations here do not differ from those in Chapter S7, but it is of interest to see them carried out in a different framework.

We may also use this approach to prove that the determinant of $f(x,y)$, which is $AC - \frac{B^2}{4}$, is invariant not only under a rotation, but also under any isometry which leaves O fixed. For this we use the fact that

$$\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right| = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|\left|\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right| ,$$

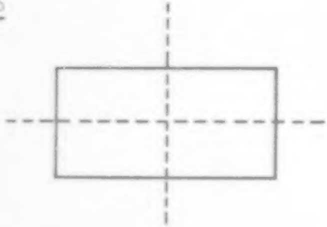
which was shown in Problem 2, Exercises S10-5b. Thus if M is the matrix of such an isometry, we have

$$A'C' - \frac{B'^2}{4} = \left|\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right| \cdot \left|\begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix}\right| \cdot \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

$$= AC - \frac{B^2}{4} \quad \text{since} \quad ad - bc = \pm 1 .$$

Exercises S10-6

1. Describe (in terms of reflections alone) the isometries of the plane which in addition carry the outline of a given rectangle into itself,



I-1. reflection in y axis : $(x,y) \rightarrow (-x,y)$

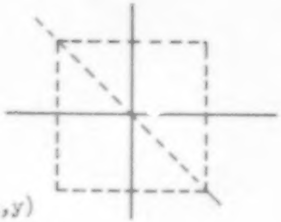
I-2. reflection in x axis : $(x,y) \rightarrow (x,-y)$

I-3. reflection in the origin $(x,y) \rightarrow (-x,-y)$

I-4. identity $(x,y) \rightarrow (x,y)$

I-1 followed I-2 has the same result as I-3.

2. Describe (in terms of reflections alone) the isometries of the plane which in addition carry the outline of a given square into itself,



I-1. reflection in the x-axis (x,y) — (-x,y)

I-2. reflection in the y-axis (x,y) — (x,-y)

I-3. reflection in the origin (x,y) — (-x,-y)

I-4. identity (x,y) — (x,y)

and in addition

I-5. reflection in the 45° line (x,y) — (y,x)

I-6. reflection in the 135° line (x,y) — (-y,-x)

3. Describe the isometries of 3-space which in addition carry a given cube into itself.